

Progressions for the Common Core State Standards in Mathematics

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For discussion of the Progressions and related topics, see the Mathematical Musings blog: <http://mathematicalmusings.org>.

Number and Operations—Fractions, 3–5

Overview

The treatment of fractions in the Standards emphasizes two features: the idea that a fraction is a number and connections with previous learning.

Fractions in the Standards In the Standards, the word “fraction” is used to refer to a type of number. That number can be expressed in different ways. It can be written in the form numerator over denominator (“in fraction notation” or “as a fraction” in conventional terminology), or in decimal notation (“as a decimal”), or—if it is greater than 1—in the form whole number followed by a number less than 1 written as a fraction (“as a mixed number”). Thus, in Grades 3–5, $\frac{7}{5}$, 1.4, and $1\frac{2}{5}$ are all considered fractions, and, in later grades, rational numbers. Expectations for computations with fractions appear in the domains of Number and Operations—Fractions, Number and Operations in Base Ten, and the Number System.

To achieve the expectations of the Standards, students need to be able to transform and use numerical—and later—symbolic expressions, including expressions for numbers. For example, in order to get the information they need or to understand correspondences between different approaches to the same problem or different representations for the same situation (MP.1), students may need to draw on their understanding of different representations for a given number. Transforming different expressions for the same number includes the skills traditionally labeled “conversion,” “reduction,” and “simplification,” but these are not treated as separate topics in the Standards. Choosing a convenient form for the purpose at hand is an important skill (MP.5), as is the fundamental understanding of equivalence of forms. Thus, $\frac{7}{5}$, 1.4, $1\frac{4}{10}$, and $1\frac{2}{5}$ are all considered acceptable expressions for the same number, although their convenience for a given purpose is likely to vary.

Building on work in earlier grades and other domains Students' work with fractions, visual representations of fractions, and operations on fractions builds on their earlier work in the domains of number, geometry, and measurement.

Units and superordinate units. First and second graders work with a variety of units and “units of units.” In learning about base-ten notation, first graders learn to think of a ten as a unit composed of 10 ones, and think of numbers in terms of these units, e.g., “20 is 2 tens” and “34 is 3 tens and 4 ones.” Second graders learn to think of a hundred as a unit composed of 10 tens as well as of 100 ones.

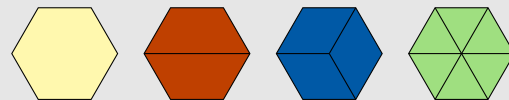
In geometry, students compose shapes. For example, first graders might put two congruent isosceles triangles together with the explicit purpose of making a rhombus. In this way, they learn to perceive a composite shape as a unit—a single new shape, e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles. Working with pattern blocks, they may build the same shape, such as a regular hexagon, from different parts, two trapezoids, three rhombuses, or six equilateral triangles.

Fraction language and subordinate units. First and second graders use fraction language to describe partitions of simple shapes into equal shares—halves, fourths, and quarters in Grade 1, extending to thirds in Grade 2.

When measuring length in Grade 3, students begin to use rulers with tick marks that indicate halves and fourths of an inch.^{3.MD.4} They are introduced to fraction notation, and their use of fractions and fraction language expands. For instance, when working with pattern blocks, a third grader might use the notation $\frac{1}{3}$ in describing a rhombus block as being one-third of a hexagon block.^{3.G.2} In the domain of number and operations, use of “half,” “fourth,” and “third” is extended to include unit fractions, that is, fractions which represent one share of a partition of 1 into equal shares. A fraction is composed of like subordinate units, e.g., $\frac{3}{4}$ is composed of 3 fourths, just as 30 is composed of 3 tens.^{3.NF.1} In Grade 3, expectations are limited to fractions with denominators 2, 3, 4, 6, and 8, allowing students to reason directly from the meaning of fraction about small fractions (i.e., fractions close to or less than 1) by folding strips of paper[•] or working with diagrams.

Diagrams. Diagrams used in work with fractions are of several types. Diagrams without numerical labels represent a whole as a two-dimensional region and a fraction as one or more equal parts of the region.[•] Use of these diagrams builds on students' work in composing and decomposing geometrical shapes, e.g., seeing a square as composed of four identical rectangles. Tape diagrams (which may be with or without numerical labels) can also represent equal parts of a whole, as well as operations on fractions. Because they represent numbers or quantities as lengths of “tape,” they tend to be less complex geometrically than area representations and may

Describing pattern block relationships at different grades



Grade 1 students might say, “A red block is half of a yellow block” and “Three blue blocks make one yellow block.”

Grade 2 students might say, “A blue block is a third of a yellow block.”

Grade 3 students begin to use notation such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{2}{3}$, and $\frac{3}{3}$ in describing the relationships of the pattern blocks.

Grade 4 students might use 0.5.

Grade 7 students might use $0.3333\dots$ or $0.\overline{3}$ (see the Number System Progression).

3.MD.4 Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

3.G.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

3.NF.1 Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$.

• These are sometimes called “fraction strips,” but in the Standards “fraction strip” is used as a synonym for “tape diagram.” The relationships shown as tape diagrams in this progression might instead or also be shown with paper strips. In later grades, however, replacing tape diagrams by paper strips may become awkward or unworkable (see the Ratios and Proportional Relationships Progression).

• These diagrams are sometimes known as “area models.” Diagrams that show rectangular regions with numerical labels are also known as “area models.” To avoid ambiguity, the former are called “area representations” in this progression.

also have the advantage of being familiar to students from work in earlier grades (see the Operations and Algebraic Thinking Progression). Other diagrams with numerical labels—number line diagrams and area models—are used to represent one or more fractions as well as relationships such as equivalence, sum or difference (number line diagram), and product or quotient (area model). Students' work with number line diagrams and area models is an abstraction and generalization of their work with length and area measurement.

Further abstractions are the notion of a number line as an infinite ruler, and, in Grade 6, the notion of a coordinate plane as an infinite two-dimensional address system. •

Length measurement and number line diagrams. Use of number line diagrams to represent fractions begins in Grade 3, building on work with measurement in Grades 1 and 2.

In Grade 1, students learn to lay physical length-units such as centimeter or inch manipulatives end-to-end and count them to measure a length.

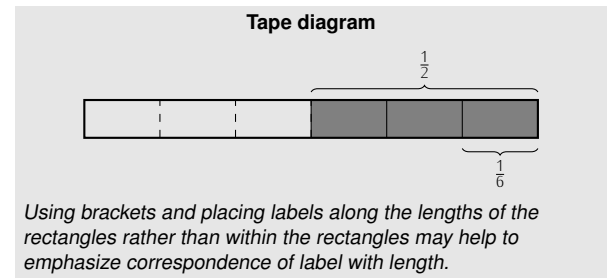
In Grade 2, students make measurements with physical length-units and rulers. They learn about the inverse relationship between the size of a length-unit and the number of length-units required to cover a given distance.^{2.MD.2}

In learning about length measurement, they develop understandings that they will use with number line diagrams:

- *length-unit iteration.* E.g., not leaving space between successive length-units;
- *accumulation of distance.* E.g., counting “eight” when placing the last length-unit means the space covered by 8 length-units, rather than just the eighth length-unit;
- *alignment of zero-point.* Correct alignment of the zero-point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;
- *meaning of numerals on the ruler.* The numerals indicate the number of length units so far;
- *connecting measurement with physical units and with a ruler.* Measuring by laying physical units end-to-end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.

These correspond to analogous conventions for number line diagrams. In particular, the unit of measurement on a ruler corresponds to the *unit interval* (length from 0 to 1) on a number line diagram. Students use number line diagrams to represent sums and differences of whole numbers in Grade 2.^{2.MD.6}

In their work with categorical and measurement data, second graders use diagrams with “count scales” that represent only whole



- This progression distinguishes between “number line” and “number line diagram,” but this is not meant to imply such distinctions should be made by teachers and students in the classroom.

2.MD.2 Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

numbers and diagrams with “measurement scales” for length unit measurements (see the Measurement and Data Progression). Both types of scales may be labeled only with whole numbers. However, subdivisions between numbers on measurement scales correspond to subdivisions of the length unit, but subdivisions between numbers on count scales may have no referent.

In Grade 3, the difference between measurement and count scales becomes more salient because students work with subdivided length-units, measuring lengths using rulers marked with halves and fourths of an inch and plotting their data.^{3.MD.4}

Area measurement and area models. Students’ work with area models begins in Grade 3. These diagrams are used in Grade 3 for single-digit multiplication and division strategies (see the Operations and Algebraic Thinking Progression), to represent multi-digit multiplication and division calculations in Grade 4 (see the Number and Operations in Base Ten Progression), and in Grades 5 and 6 to represent multiplication and division of fractions (see this progression and the Number System Progression). The distributive property is central to all of these uses.

Work with area models builds on previous work with area measurement. As with length measurement, area measurement relies on several understandings:

- *area is invariant.* Congruent figures enclose regions with equal areas;
- *area is additive.* The area of the union of two regions that overlap only at their boundaries is the sum of their areas;
- *area-unit tiling.* Area is measured by tiling a region with a two-dimensional area-unit (such as a square or rectangle) and parts of the unit, without gaps or overlaps.

Perceiving a region as tiled by an area-unit relies on *spatial structuring*. For example, second graders learn to see how a rectangular region can be partitioned as an array of squares.^{2.G.2} Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. For further discussion, see the K–6 Geometry Progression.

Addition and subtraction. In Grades 4 and 5, students learn about operations on fractions, extending the meanings of the operations on whole numbers. For addition and subtraction, these meanings arise from the Add To, Take From, Put Together/Take Apart, and Compare problem types and are established before Grade 3.[•]

In Grade 4, students compute sums and differences, mainly of fractions and mixed numbers with like denominators. In Grade 5, students use their understanding of equivalent fractions to compute sums and differences of fractions with unlike denominators.

3.MD.4 Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

2.G.2 Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

• For descriptions and examples of these problem types, see the Overview of K–2 in the Operations and Algebraic Thinking Progression.

Multiplication. The concept of multiplication begins in Grade 3 with an entirely discrete notion of “equal groups.”^{3.OA.1} By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion “times as much.”^{4.OA.1} This notion has more affinity to continuous quantities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as $\frac{3}{4}$ cup of flour.^{4.NF.4, 4.MD.2} By Grade 5, when students multiply fractions in general,^{5.NF.4} products can be larger or smaller than either factor, and multiplication can be seen as an operation that “stretches or shrinks” by a scale factor.^{5.NF.5}

Grade 3 work with whole-number multiplication and division focuses on two problem types, Equal Groups and Arrays. (For descriptions of these problem types and examples that involve discrete attributes, see the Grade 3 section of the Operations and Algebraic Thinking Progression. For examples with continuous attributes, see the Geometric Measurement Progression. Both illustrate measurement (quotitive) and sharing (partitive) interpretations of division.)

Initially, problems involve multiplicands that represent discrete attributes (e.g., cardinality). Later problems involve continuous attributes (e.g., length). For example, problems of the Equal Groups type involve situations such as:

- There are 3 bags with 4 plums in each bag. How many plums are there in all?

and, in the domain of measurement:

- You need 3 lengths of string, each 4 feet long. How much string will you need altogether?

Both of these problems are about 3 groups of four things each—3 fours—in which the group of four can be seen as a whole (1 bag or 1 length of string) or as a composite of units (4 plums or 4 feet). In the United States, the multiplication expression for 3 groups of four is usually written as 3×4 , with the multiplier first. (This convention is used in this progression. However, as discussed in the Operations and Algebraic Thinking Progression, some students may write 4×3 and it is useful to discuss the different interpretations in connection with the commutative property.)

In Grade 4, problem types for whole-number multiplication and division expand to include Multiplicative Compare with whole numbers. In this grade, Equal Groups and Arrays extend to include problems that involve multiplying a fraction by a whole number. For example, problems of the Equal Groups type might be:

- You need 3 lengths of string, each $\frac{1}{4}$ foot long. How much string will you need altogether?
- You need 3 lengths of string, each $\frac{5}{4}$ feet long. How much string will you need altogether?

3.OA.1 Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

4.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5.NF.5 Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Like the two previous problems, these two problems are about objects that can be seen as wholes (1 length of string) or in terms of units. However, instead of being composed of units (feet), they are composed of subordinate units ($\frac{1}{4}$ -feet).

In Grade 5, students connect fractions with division, understanding numerical instances of $\frac{a}{b} = a \div b$ for whole numbers a and b , with b not equal to zero (MP.8).^{5.NF.3} With this understanding, students see, for example, that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any number by a fraction. Problem types for multiplication expand to include Multiplicative Compare with unit fraction language, e.g., “one third as much as,” and students solve problems that involve multiplying by a fraction. For example, a problem of the Equal Groups type might be:

- You need $\frac{1}{3}$ of a length of string that is $2\frac{1}{4}$ feet long. How much string will you need altogether?

Measurement conversion. At Grades 4 and 5, expectations for conversion of measurements parallel expectations for multiplication by whole numbers and by fractions. In 4.MD.1, the emphasis is on “times as much” or “times as many,” conversions that involve viewing a larger unit as superordinate to a smaller unit and multiplying the number of larger units by a whole number to find the number of smaller units. For example, conversion from feet to inches involves viewing a foot as superordinate to an inch, e.g., viewing a foot as 12 inches or as 12 times as long as an inch, so a measurement in inches is 12 times what it is in feet. In 5.MD.1, conversions also involve viewing a smaller unit as subordinate to a larger one, e.g., an inch is $\frac{1}{12}$ foot, so a measurement in feet is $\frac{1}{12}$ times what it is in inches and conversions require multiplication by a fraction (5.NF.4).

Division. Using their understanding of division of whole numbers and multiplication of fractions, students in Grade 5 solve problems that involve dividing a whole number by a unit fraction or a unit fraction by a whole number. In Grade 6, they extend their work to problems that involve dividing a fraction by a fraction (see the Number System Progression).

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

- See the Grade 4 section of the Operations and Algebraic Thinking Progression for discussion of linguistic aspects of “as much” and related formulations for Multiplicative Compare problems.

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

5.MD.1 Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Grade 3

The meaning of fractions and fraction notation In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.^{2.G.3} In Grade 3, they start to develop a more general concept of fraction, building on the idea of partitioning a whole into equal parts and expressing the number of parts symbolically, using fraction notation. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision. In Grade 4, this is extended to include wholes that are collections of objects.

Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is what you get by putting 3 of the $\frac{1}{4}$ s together.^{3.NF.1} They read any fraction this way. In particular there is no need to introduce “proper fractions” and “improper fractions” initially; $\frac{5}{3}$ is what you get by combining 5 parts when a whole is partitioned into 3 equal parts.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP.6):

- Specifying the whole.
- Explaining what is meant by “equal parts.”

Initially, students can use an intuitive notion of congruence (“same size and same shape” or “matches exactly”) to explain why the parts are equal, e.g., when they partition a square into four equal squares or four equal rectangles.^{3.G.2}

Students come to understand a more precise meaning for “equal parts” as “parts with equal measurements.” For example, when a ruler is partitioned into halves or quarters of an inch,^{3.MD.4} students see that each subdivision has the same length. Labeling the ruler marks can help students understand rulers marked with halves and fourths, but not labeled with these fractions. Analyzing area models, students reason about the area of a shaded region to decide what fraction of the whole it represents (MP.3).

The goal is for students to see unit fractions as basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers. Just as every whole number can be obtained by combining ones, every fraction can be obtained by combining copies of one unit fraction.

2.G.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

3.NF.1 Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$.

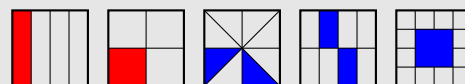
The importance of specifying the whole



Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

3.G.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

Area representations of $\frac{1}{4}$



In each representation, the square is the whole. The two squares on the left are partitioned into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a partition of the square into four parts of the same shape and size.

3.MD.4 Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

The number line and number line diagrams On a number line diagram, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler with the unit interval as the unit of measurement.

To construct a unit fraction on a number line diagram, e.g., $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They determine the location of the number $\frac{1}{3}$ by marking off this length from 0, and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.^{3.NF.2}

Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, initially they use other representations such as area representations, strips of paper, and tape diagrams. These, like number line diagrams, can be subdivided, representing an important aspect of fractions.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the unit of measurement, namely the interval from 0 to $\frac{1}{3}$.

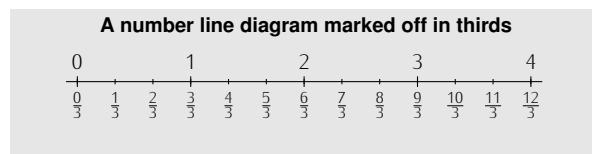
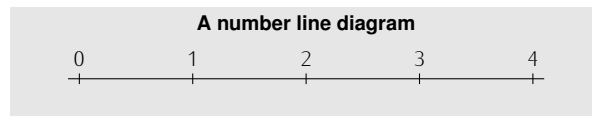
Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions, in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are therefore equal; that is, they are *equivalent fractions*. For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$ and also to $\frac{3}{6}$. Students can also use tape diagrams to see fraction equivalence.^{3.NF.3}

In particular, students in Grade 3 see whole numbers as fractions, recognizing, for example, that the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc. so that^{3.NF.3c}

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \dots$$

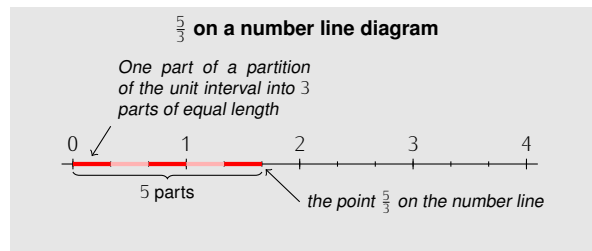
Of particular importance are the ways of writing 1 as a fraction:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \dots$$



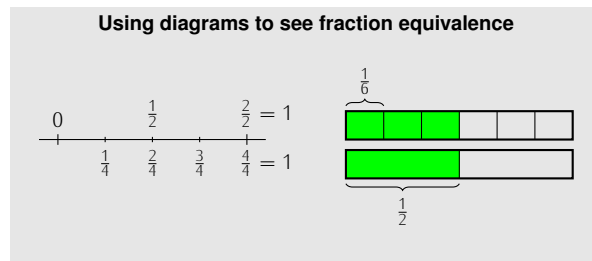
3.NF.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

- Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line.
- Represent a fraction $\frac{a}{b}$ on a number line diagram by marking off a lengths $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line.



3.NF.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

- Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
- Recognize and generate simple equivalent fractions, e.g., $\frac{1}{2} = \frac{2}{4}$, $\frac{4}{6} = \frac{2}{3}$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.
- Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.



Comparing fractions Previously, in Grade 2, students compared lengths using a standard unit of measurement.^{2.MD.3} In Grade 3, they build on this idea to compare fractions with the same denominator. They see that for two fractions that have the same denominator, the underlying unit fractions are the same size, so the fraction with the greater numerator is greater because it is made of more unit fractions. For example, on the number line the segment from 0 to $\frac{3}{4}$ is shorter than the segment from 0 to $\frac{5}{4}$ because it is 3 fourths long as opposed to 5 fourths long. Therefore $\frac{3}{4} < \frac{5}{4}$.^{3.NF.3d}

In Grade 2, students gained experience with the inverse relationship between the size of a physical length-unit and the number of length-units required to cover a given distance.^{2.MD.2} Grade 3 students see that for unit fractions, the one with the larger denominator is smaller, by reasoning, for example, that in order for more (identical) pieces to make the same whole, the pieces must be smaller. From this they reason that for two fractions that have the same numerator, the fraction with the smaller denominator is greater. For example, $\frac{2}{5} > \frac{2}{7}$, because $\frac{1}{7} < \frac{1}{5}$, so 2 lengths of $\frac{1}{7}$ is less than 2 lengths of $\frac{1}{5}$.

As with equivalence of fractions, it is important in comparing fractions to make sure that each fraction refers to the same whole.

As students move towards understanding fractions as points on the number line, they develop an understanding of order in terms of position. Given two fractions—thus two points on the number line—the one to the left is said to be smaller and the one to the right is said to be larger. This understanding of order as position will become important in Grade 6 when students start working with negative numbers.

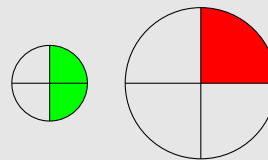
2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

3.NF.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

- d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.

2.MD.2 Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

The importance of referring to the same whole when comparing fractions



A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction equal to the original fraction. This property forms the basis for much of the other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

Equivalent fractions Students can use area representations, strips of paper, tape diagrams, and number line diagrams to reason about equivalence.^{4.NF.1} They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, n , corresponds to partitioning each piece of the diagram into n smaller equal pieces (MP.1). Each region or length that represents a unit fraction (a subordinate unit) is partitioned into n smaller regions or lengths, each of which represents a unit fraction (a subordinate unit of a subordinate unit). The whole has then been partitioned into n times as many pieces, and there are n times as many smaller unit fraction pieces as in the original fraction.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction as a point on the number line.

The fundamental property can be presented in terms of division, as in, e.g.

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}.$$

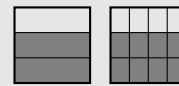
Because the equations $28 \div 4 = 7$ and $36 \div 4 = 9$ tell us that $28 = 4 \times 7$ and $36 = 4 \times 9$, this is the fundamental property in disguise:

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

It is possible to over-emphasize the importance of simplifying fractions in this way. There is no mathematical reason why fractions must be written in simplified form, although it may be convenient to do so in some cases.

4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area representation to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$



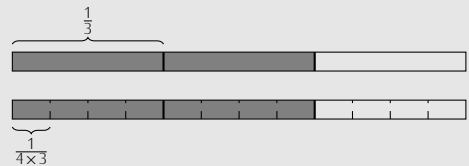
The whole is the square. On the left, the square is partitioned into 3 rectangles of equal area (3 thirds). The shaded region is 2 of these thirds, so represents $\frac{2}{3}$.

To get the figure on the right, each of the 3 rectangles has been partitioned into 4 smaller rectangles of equal area.

Viewed in terms of rows, this makes 3 rows of 4 small rectangles, so the square is now partitioned into 3×4 equal pieces (12 twelfths). The shaded area is 2×4 of these twelfths, so represents $\frac{2 \times 4}{3 \times 4}$.

Viewed in terms of columns, this makes 4 columns of 3 small rectangles, so the square is now partitioned into 4×3 equal pieces (12 twelfths). The shaded area is 4×2 of these twelfths, so represents $\frac{4 \times 2}{4 \times 3}$.

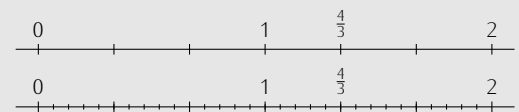
Using a tape diagram to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$



The whole is the tape. In the top diagram, the tape is partitioned into 3 equal pieces, thus each piece represents $\frac{1}{3}$ and the shaded section represents $\frac{2}{3}$. Each section of the top diagram is partitioned into four equal pieces to produce the bottom diagram.

In the bottom diagram, the tape is partitioned into 4×3 equal pieces, thus each piece represents $\frac{1}{4 \times 3}$ and the shaded section represents $\frac{4 \times 2}{4 \times 3}$.

Using a number line diagram to show that $\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$



$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5 \times 3}$. Partition each interval of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5 \times 4}{5 \times 3} = \frac{20}{15}$.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.^{4.NF.2} For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they rewrite both fractions:

$$\frac{12 \times 5}{12 \times 8} = \frac{60}{96} \quad \text{and} \quad \frac{7 \times 8}{12 \times 8} = \frac{56}{96}.$$

Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is therefore to the left of 1) but $\frac{13}{12}$ is greater than 1 (and is therefore to the right of 1).

Grade 4 students who have learned about fraction multiplication can see equivalence as “multiplying by 1”:

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}$$

However, although a useful mnemonic device, this does not constitute a valid argument at this grade, since students have not yet learned fraction multiplication.

Adding and subtracting fractions The meaning of addition is the same for both fractions and whole numbers, even though algorithms for calculating their sums can be different. Just as the sum of 4 and 7 can be interpreted as the length of the segment obtained by putting together two segments of lengths 4 and 7, so the sum of $\frac{2}{3}$ and $\frac{8}{5}$ can be interpreted as the length of the segment obtained by putting together two segments of length $\frac{2}{3}$ and $\frac{8}{5}$. It is not necessary to know how much $\frac{2}{3} + \frac{8}{5}$ is exactly in order to know what the sum means. This is analogous to interpreting 51×78 as 51 groups of 78, without necessarily knowing its exact value.

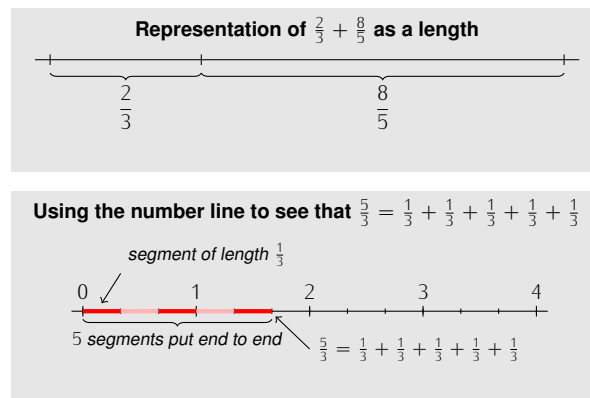
This simple understanding of addition as putting together allows students to see in a new light the way fractions are composed of unit fractions. Number line diagrams, the same type of diagrams that students used in Grade 3 to see a fraction as a point on the number line, allows them to see a fraction as a sum of unit fractions. Just as $5 = 1 + 1 + 1 + 1 + 1$, so

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

because $\frac{5}{3}$ is the total length of 5 thirds.^{4.NF.3}

Armed with this insight, students decompose and compose fractions with the same denominator. They add fractions with the same

4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $\frac{1}{2}$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.



4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

- Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.
- Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.
- Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

denominator.^{4.NF.3c} Here, equations are used to describe approaches that might also be shown with diagrams:

$$\begin{aligned} \frac{7}{5} + \frac{4}{5} &= \overbrace{\frac{1}{5} + \cdots + \frac{1}{5}}^7 + \overbrace{\frac{1}{5} + \cdots + \frac{1}{5}}^4 \\ &= \frac{\overbrace{1 + 1 + \cdots + 1}^{7+4}}{5} \\ &= \frac{7+4}{5}. \end{aligned}$$

Using the understanding gained from work with whole numbers of the relationship between addition and subtraction, they also subtract fractions with the same denominator. For example, to subtract $\frac{5}{6}$ from $\frac{17}{6}$, they decompose

$$\frac{17}{6} = \frac{12}{6} + \frac{5}{6}, \quad \text{so} \quad \frac{17}{6} - \frac{5}{6} = \frac{17-5}{6} = \frac{12}{6} = 2.$$

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators. • Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as addition.

Similarly, converting an improper fraction to a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.^{4.NF.3b} Students can draw on their knowledge from Grade 3 of whole numbers as fractions. For example, knowing that $1 = \frac{3}{3}$, they see

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

Calculations with mixed numbers provide opportunities for students to compare approaches and justify steps in their computations (MP.3). For example, $2\frac{1}{3} - \frac{2}{3}$ may be calculated in a variety of ways. Here, equations with parentheses are used to describe three approaches that students might take but not necessarily their steps or symbolism. •

Converting the 2 to $\frac{6}{3}$:

$$\begin{aligned} 2\frac{1}{3} - \frac{2}{3} &= \left(\frac{6}{3} + \frac{1}{3}\right) - \frac{2}{3} \\ &= \frac{7}{3} - \frac{2}{3} = \frac{5}{3}. \end{aligned}$$

- A mixed number is a number written as a whole number plus a fraction smaller than 1, written without the + sign, e.g. $5\frac{3}{4}$ means $5 + \frac{3}{4}$ and $7\frac{1}{5}$ means $7 + \frac{1}{5}$.

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

- b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

- Use of parentheses, but not necessarily fluency with parentheses, is expected in Grade 5 (see 5.OA.1), however reading expressions with parentheses may begin earlier.

Decomposing the 2 into 1 + 1, and using the associative and commutative properties:

$$\begin{aligned} 2\frac{1}{3} - \frac{2}{3} &= \left(1 + \frac{1}{3}\right) + \left(1 - \frac{2}{3}\right) \\ &= 1 + \left(\frac{1}{3} + \frac{1}{3}\right) \\ &= 1 + \frac{2}{3} = 1\frac{2}{3}. \end{aligned}$$

Decomposing a one into 3 thirds:

$$\begin{aligned} 2\frac{1}{3} - \frac{2}{3} &= \left(1 + \frac{3}{3} + \frac{1}{3}\right) - \frac{2}{3} \\ &= 1\frac{4}{3} - \frac{2}{3} \\ &= 1\frac{2}{3}. \end{aligned}$$

The third approach is an analogue of what students learned when subtracting two-digit whole numbers in Grade 2: decomposing a unit of the minuend into smaller units (see the Number and Operations in Base Ten Progression). Instead of decomposing a ten into 10 ones as in Grade 2, a one has been decomposed into 3 thirds. The same approach of decomposing a one (this time into 10 tenths) could be used to compute $2\frac{1}{10} - \frac{2}{10}$:

$$2\frac{1}{10} - \frac{2}{10} = \left(1 + \frac{10}{10} + \frac{1}{10}\right) - \frac{2}{10} = 1\frac{11}{10} - \frac{2}{10} = 1\frac{9}{10}.$$

This approach is used in Grade 5 when such computations are carried out in decimal notation.^{5.NBT.7}

Repeated reasoning with examples that gain in complexity leads to a general method involving the Grade 4 NBT skill of finding quotients and remainders.^{4.NBT.6} For example,

$$\frac{47}{6} = \frac{(7 \times 6) + 5}{6} = \frac{7 \times 6}{6} + \frac{5}{6} = 7 + \frac{5}{6} = 7\frac{5}{6}.$$

When solving word problems students learn to attend carefully to the underlying quantities (MP.6). In an equation of the form $A + B = C$ or $A - B = C$ for a word problem, the numbers A , B , and C must all refer to the same whole, in terms of the same units.^{4.NF.3d} For example, students understand that the problem

Bill had $\frac{2}{3}$ cup of juice. He drank half of his juice. How much juice did Bill have left?

cannot be solved by computing $\frac{2}{3} - \frac{1}{2}$. Although the $\frac{2}{3}$ and “half” both refer to the same whole (the amount of juice that Bill had), the $\frac{2}{3}$ refers to the whole measured in cups, but the half refers to the amount of juice that Bill had as a unit, not measured in cups.

^{5.NBT.7} Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

^{4.NBT.6} Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

^{4.NF.3} Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

Similarly, in solving

If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{4} + \frac{1}{3}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 3×7 can be represented as the number of objects in 3 groups of 7 objects, and write this as $7 + 7 + 7$. Grade 4 students apply this understanding to fractions, seeing

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \text{ as } 5 \times \frac{1}{3}.$$

In general, they see a fraction as the numerator times the unit fraction with the same denominator,^{4.NF.4a} e.g.,

$$\frac{7}{5} = 7 \times \frac{1}{5}, \quad \frac{11}{3} = 11 \times \frac{1}{3}.$$

The same thinking, based on the analogy between fractions and whole numbers, allows students to give meaning to the product of a whole number and a fraction,^{4.NF.4b} e.g., they see

$$3 \times \frac{2}{5} \text{ as } \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}.$$

Students solve word problems involving multiplication of a fraction by a whole number.^{4.NF.4c}

If a bucket holds $2\frac{3}{4}$ gallons and 3 buckets of water fill a tank, how many gallons does the tank hold?

The answer is $3 \times 2\frac{3}{4}$, which is

$$3 \times \left(2 + \frac{3}{4}\right) = 3 \times \frac{11}{4} = \frac{33}{4} = 8\frac{1}{4}.$$

Decimal fractions and decimal notation Fractions with denominators 10 and 100, called *decimal fractions*, arise when students convert from dollars to cents,^{4.MD.2} and have a more fundamental importance, developed in Grade 5, in the base-ten system (see the Grade 5 section of the Number and Operations in Base Ten Progression). For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

Draft, August 10, 2018.

4.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

- a Understand a fraction a/b as a multiple of $1/b$.
- b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.
- c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Grade 4 students learn to add decimal fractions by converting them to fractions with the same denominator, in preparation for general fraction addition in Grade 5:^{4.NF.5}

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Fractions with denominators equal to 10, 100, etc., such as

$$\frac{27}{10}, \frac{27}{100}, \text{ etc.}$$

can be written by using a *decimal point*• as^{4.NF.6}

$$2.7, 0.27.$$

The number of digits to the right of the decimal point indicates the number of zeros in the denominator, so that $2.70 = \frac{270}{100}$ and $2.7 = \frac{27}{10}$. Students use their ability to convert fractions to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

Reflecting these understandings, there are several ways to read decimals aloud. For example, 0.15 can be read aloud as “1 tenth and 5 hundredths” or “15 hundredths,” just as 1,500 is sometimes read “15 hundred” or “1 thousand, 5 hundred.” (Mathematicians and scientists often read 0.15 aloud as “zero point one five” or “point one five.”) Similarly, 150 is read “one hundred and fifty” or “a hundred fifty” and understood as 15 tens, as 10 tens and 5 tens, and as 100 + 50.

Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of them as $\frac{20}{100}$ and $\frac{9}{100}$ and see that $0.20 > 0.09$ because^{4.NF.7}

$$\frac{20}{100} > \frac{9}{100}.$$

The argument using the meaning of a decimal as a fraction generalizes to work with decimals in Grade 5 that have more than two digits, whereas the argument using an area representation, shown in the margin, does not. So it is useful for Grade 4 students to see such reasoning.

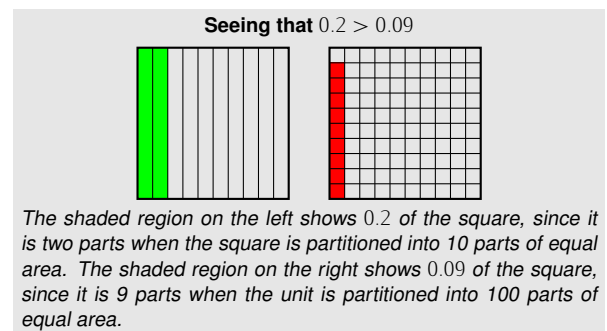
4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.⁴

⁴Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

• Decimals smaller than 1 may be written with or without a zero before the decimal point.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.



Grade 5

Adding and subtracting fractions In Grade 4, students acquire some experience in calculating sums of fractions with different denominators when they work with decimals and add fractions with denominators 10 and 100, such as

$$\frac{2}{10} + \frac{7}{100} = \frac{20}{100} + \frac{7}{100} = \frac{27}{100}.$$

Note that this is a situation where one denominator is a divisor of the other, so that only one fraction has to be changed. Students might have encountered similar situations, for example using a strip of paper or a tape diagram to reason that

$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

They understand the process as expressing both summands in terms of the same unit fraction so that they can be added. Grade 5 students extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.^{5.NF.1} For example, in calculating $\frac{2}{3} + \frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is partitioned into four equal parts, and if each fourth in $\frac{5}{4}$ is partitioned into three equal parts, then each fraction will be a sum of unit fractions with denominator $3 \times 4 = 4 \times 3 = 12$:

$$\frac{2}{3} + \frac{5}{4} = \frac{2 \times 4}{3 \times 4} + \frac{5 \times 3}{4 \times 3} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}.$$

In general, two fractions can be added by partitioning the unit fractions in one into the number of equal parts determined by the denominator of the other:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{a \times d + b \times c}{b \times d}.$$

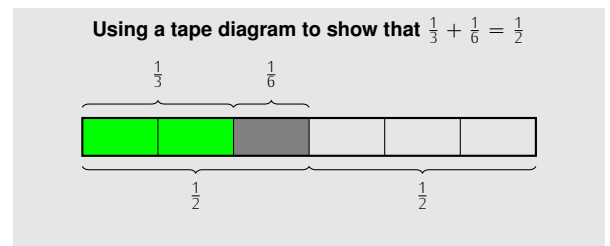
It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.^{5.NF.2} For example in the problem

Ludmilla and Lazarus each have some lemons. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ cup from hers and Lazarus squeezes $\frac{2}{5}$ cup from his. How much lemon juice do they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5} < \frac{1}{2}$. They calculate $\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$ by noticing that $\frac{3}{7} < \frac{1}{2}$.

Draft, August 10, 2018.



5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

Multiplying and dividing fractions In Grade 4, students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b , with b not equal to zero.^{5.NF.3} They can explain this using the sharing (partitive) interpretation of division (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, calculating $50 \times \frac{1}{9} = \frac{50}{9}$ so that each person gets $\frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as “third of” to describe one part when a whole is partitioned into three equal parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any number by a fraction.^{5.NF.4a} Just as

$\frac{1}{3} \times 5$ is 1 part when 5 is partitioned into 3 equal parts,

so

$\frac{4}{3} \times 5$ is 4 parts when 5 is partitioned into 3 equal parts.

Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d},$$

for whole numbers a, b, c, d , with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 people:
 $5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

If you share 5 objects equally among 3 people, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus, each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object; $5 \times \frac{1}{3} = \frac{5}{3}$, so each share is $\frac{5}{3}$ of an object.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

- a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a tape diagram to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

One half partitioned in 3 equal parts. Each of the 3 shaded pieces is $\frac{1}{3}$ of $\frac{1}{2}$, which is $\frac{1}{3} \times \frac{1}{2}$.

Other half partitioned in 3 equal parts.

There are 6 equal pieces, so each is $\frac{1}{6}$, as well as $\frac{1}{3} \times \frac{1}{2}$.

Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2}$

Partition each length of $\frac{1}{2}$ in 3 equal parts. Each is $\frac{1}{3} \times \frac{1}{2}$, which is $\frac{1}{3 \times 2}$.

The length of 2 pieces is $2 \times \frac{1}{3 \times 2}$.

There are 5 lengths of $\frac{1}{2}$. Taking 2 pieces from each length of $\frac{1}{2}$ makes a length of $5 \times (2 \times \frac{1}{3 \times 2})$, which is $\frac{2 \times 5}{3 \times 2}$.

rather recognize numerical instances from reasoning repeatedly from many examples (MP.8), using strips of paper, tape diagrams, and number line diagrams.

Having established a meaning for the product of two fractions and an understanding of how to calculate such products, students use concepts of area measurement from Grade 3^{3.MD.5} to see that the method that they used to find areas of rectangles with whole-number side lengths in Grade 3^{3.MD.7a} can be extended to rectangles with fractional side lengths.^{5.NF.4b} Instead of using a unit square with a side length of 1 inch or 1 centimeter, fifth graders use a unit square with a side length that is a fractional unit. For example, a $\frac{5}{3}$ by $\frac{1}{2}$ rectangle can be tiled by 30 unit squares with side length $\frac{1}{6}$. Because 36 of these of these unit squares tile a 1 by 1 square, each has area $\frac{1}{36}$. So the area of the rectangle is 30 thirty-sixths, which is $\frac{5}{3} \times \frac{1}{2}$, the product of the side lengths.

Students can use similar reasoning with other tilings of a 1-by-1 square. For example, when working with a rectangle that has fractional side lengths, students can see it as tiled by copies of a smaller rectangle whose sides are the corresponding unit fractions. Because 12 copies of the smaller rectangle tile a 1-by-1 square, each copy has area $\frac{1}{12}$ (see illustration in the margin).

Students also understand fraction multiplication by creating story problems. For example, to explain

$$\frac{2}{3} \times 4 = \frac{8}{3},$$

they might say

Ron and Hermione have 4 pounds of Bertie Bott’s Every Flavour Beans. They decide to share them 3 ways, saving one share for Harry. How many pounds of beans do Ron and Hermione get?

In multiplication calculations, the distributive property may be shown symbolically or—because the area of a rectangle is the product of its side lengths—with an area model. Here, it is used in a variation of a word problem from the Grade 4 section.

If a bucket holds $2\frac{3}{4}$ gallons and 43 buckets of water fill a tank, how many gallons does the tank hold?

The answer is $43 \times 2\frac{3}{4}$, which is

$$\begin{aligned} 43 \times \left(2 + \frac{3}{4}\right) &= 43 \times 2 + 43 \times \frac{3}{4} \\ &= 86 + \left(40 \times \frac{3}{4}\right) + \left(3 \times \frac{3}{4}\right) \\ &= 86 + 30 + \frac{9}{4} \\ &= 118\frac{1}{4}. \end{aligned}$$

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

- a A square with side length 1 unit, called “a unit square,” is said to have “one square unit” of area, and can be used to measure area.
- b A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.7 Relate area to the operations of multiplication and addition.

- a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

- b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3 \times 5}{4 \times 3}$

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high fit in a 1-by-1 square, $\frac{1}{4} \times \frac{1}{3} = \frac{1}{4 \times 3}$.

The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4 \times 3}$, so has area $\frac{3 \times 5}{4 \times 3}$.

Using an area model to calculate $43 \times 2\frac{3}{4}$

Using the relationship between division and multiplication, students start working with quotients that have unit fractions. Having seen that dividing a whole number by a whole number, e.g., $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that^{5.NF.7a}

$$\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}.$$

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be 3×6 in 3, and so^{5.NF.7b}

$$3 \div \frac{1}{6} = 3 \times 6 = 18.$$

Students use story problems to make sense of division:^{5.NF.7c}

How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$ -cup servings are in 2 cups of raisins?

Students attend carefully to the underlying quantities when solving problems. For example, if $\frac{1}{2}$ of a fund-raiser's funds were raised by the 6th grade, and if $\frac{1}{3}$ of the 6th grade's funds were raised by Ms. Wilkin's class, then $\frac{1}{3} \times \frac{1}{2}$ gives the fraction of the fund-raiser's funds that Ms. Wilkin's class raised, but it does not tell us how much money Ms. Wilkin's class raised.^{5.NF.6}

Multiplication as scaling In preparation for Grade 6 work with ratios and proportional relationships, students learn to see products such as 5×3 or $\frac{1}{2} \times 3$ as expressions that can be interpreted as an amount, 3, and a scaling factor, 5 or $\frac{1}{2}$. Thus, in addition to knowing that $5 \times 3 = 15$, they can also say that 5×3 is 5 times as big as 3, without evaluating the product. Likewise, they see $\frac{1}{2} \times 3$ as half the size of 3.^{5.NF.5a}

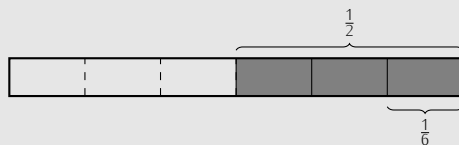
The understanding of multiplication as scaling is an important opportunity for students to reason abstractly (MP.2). Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a price is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when a price is multiplied by $\frac{1}{2}$, for example.^{5.NF.5b}

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 11.

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

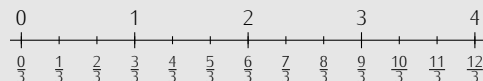
- Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.
- Interpret division of a whole number by a unit fraction, and compute such quotients.
- Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

Division of a unit fraction by a whole number: $\frac{1}{2} \div 3$



Reasoning with a tape diagram using the sharing interpretation of division: the tape is the whole and the shaded length is $\frac{1}{2}$ of the whole. If the shaded length is partitioned into 3 equal parts, then 2×3 of those parts compose the whole, so $\frac{1}{2} \div 3 = \frac{1}{2 \times 3} = \frac{1}{6}$.

Division of a whole number by a unit fraction: $4 \div \frac{1}{3}$



Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.5 Interpret multiplication as scaling (resizing), by:

- Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
- Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Where this progression is heading

In Grade 5, students interpreted a fraction as division of the numerator by the denominator, e.g., they saw that $5 \div 3 = \frac{5}{3}$. In Grade 6, students see whole numbers and fractions as part of the system of rational numbers, understanding order, magnitude, and absolute value in terms of the number line. In Grade 7, students use properties of operations and their understanding of operations on fractions to extend those operations to rational numbers. Their new understanding of division allows students to extend their use of fraction notation from non-negative rational numbers to all rational numbers, e.g., $\frac{-3}{4} = -3 \div 4$ and $\frac{\frac{2}{3}}{-\frac{1}{2}} = \frac{2}{3} \div -\frac{1}{2}$ (see the Number System Progression).

Work with fractions and multiplication is a building block for work with ratios. In Grades 6 and 7, students use their understanding of wholes and parts to reason about ratios of two quantities, making and analyzing tables of equivalent ratios, and graphing pairs from these tables in the coordinate plane. These tables and graphs represent proportional relationships, which students see as functions in Grade 8.

Understanding of multiplication as scaling is extended in work with ratios (see the Ratios and Proportional Relationships Progression) and in work with scale drawings (see the 7–8 Geometry Progression). Students' understanding of scaling is further extended when they work with similarity and dilations of the plane, using physical models, transparencies, or geometry software in Grade 8, and using properties of dilations in high school (see the high school Geometry Progression).

- Note that in the Standards, “fraction” and “ratio” refer to different concepts and that different notation is used with each. For example, $\frac{3}{2}$ is not used to represent $3 : 2$. Equivalence for fractions is denoted with the equal sign, e.g., $\frac{3}{2} = \frac{6}{4}$, but the equal sign is not used to denote the equivalence of two pairs of numerical measurements that are in the same ratio.