Front Matter for Progressions for the Common Core State Standards in Mathematics (draft)

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For updates and more information about the Progressions, see `http://ime.math.arizona.edu/progressions`.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: `http://commoncoretools.me`.

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*This list includes Progressions that currently exist in draft form as well as planned Progressions.

Preface for the Draft Progressions

The Common Core State Standards in mathematics began with progressions: narrative documents describing the progression of a topic across a number of grade levels, informed both by educational research and the structure of mathematics. These documents were then sliced into grade level standards. From that point on the work focused on refining and revising the grade level standards, thus, the early drafts of the progressions documents do not correspond to the 2010 Standards.

The Progressions for the Common Core State Standards are updated versions of those early progressions drafts, revised and edited to correspond with the Standards by members of the original Progressions work team, together with other mathematicians and education researchers not involved in the initial writing. They note key connections among standards, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics.

Audience The Progressions are intended to inform teacher preparation and professional development, curriculum organization, and textbook content. Thus, their audience includes teachers and anyone involved with schools, teacher education, test development, or curriculum development. Members of this audience may require some guidance in working their way through parts of the mathematics in the draft Progressions (and perhaps also in the final version of the Progressions). As with any written mathematics, understanding the Progressions may take time and discussion with others.

Revision of the draft Progressions will be informed by comments and discussion at `http://commoncoretools.me`, The Tools for the Common Core blog. This blog is a venue for discussion of the Standards as well as the draft Progressions and is maintained by lead Standards writer Bill McCallum.

Scope Because they note key connections among standards and topics, the Progressions offer some guidance but not complete guidance about how topics might be sequenced and approached across and within grades. In this respect, the Progressions are an intermediate step between the Standards and a teachers manual for a grade-level textbook—a type of document that is uncommon in the United States.

Other sources of information Another important source of information about the Standards and their implications for curriculum is the *Publishers' Criteria for the Common Core State Standards for Mathematics*, available at `www.corestandards.org`. In addition to giving criteria for evaluating K–12 curriculum materials, this document gives a brief and very useful orientation to the Standards in its short essay "The structure is the Standards."

Illustrative Mathematics illustrates the range and types of mathematical work that students experience in a faithful implementation of the Common Core State Standards. This and other ongoing projects that involve the Standards writers and support the Common Core are listed at `http://ime.math.arizona.edu/commoncore`.

Understanding Language aims to heighten awareness of the critical role that language plays in the new Common Core State Standards and Next Generation Science Standards, to synthesize knowledge, and to develop resources that help ensure teachers can meet their students' evolving linguistic needs as the new Standards are implemented. See `http://ell.stanford.edu`.

Teachers' needs for mathematical preparation and professional development in the context of the Common Core are often substantial. The Conference Board of the Mathematical Sciences report *The Mathematical Education of Teachers II* gives recommendations for preparation and professional development, and for mathematicians' involvement in teachers' mathematical education. See `www.cbmsweb.org/MET2/index.htm`.

Acknowledgements Funding from the Brookhill Foundation for the Progressions Project is gratefully acknowledged. In addition to benefiting from the comments of the reviewers who are members of the writing team, the Progressions have benefited from other comments, many of them contributed via the Tools for the Common Core blog.

Introduction

The college- and career-readiness goals of the Common Core State Standards of the Standards were informed by surveys of college faculty, studies of college readiness, studies of workplace needs, and reports and recommendations that summarize such studies.• Created to achieve these goals, the Standards are informed by the structure of mathematics as well as three areas of educational research: large-scale comparative studies, research on children's learning trajectories, and other research on cognition and learning.

References to work in these four areas are included in the "works consulted" section of the Standards document. This introduction outlines how the Standards have been shaped by each of these influences, describes the organization of the Standards, discusses how traditional topics have been reconceptualized to fit that organization, and mentions aspects of terms and usage in the Standards and the Progressions.

• These include the reports from Achieve, ACT, College Board, and American Diploma Project listed in the references for the Common Core State Standards as well as sections of reports such as the American Statistical Association's *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A PreK–12 Curriculum Framework* and the National Council on Education and the Disciplines' *Mathematics and Democracy, The Case for Quantitative Literacy*.

The structure of mathematics One aspect of the structure of mathematics is reliance on a small collection of general properties rather than a large collection of specialized properties. For example, addition of fractions in the Standards extends the meanings and properties of addition of whole numbers, applying and extending key ideas used in addition of whole numbers to addition of unit fractions, then to addition of all fractions.• As number systems expand from whole numbers to fractions in Grades 3–5, to rational numbers in Grades 6–8, to real numbers in high school, the same key ideas are used to define operations within each system.

• In elementary grades, "whole number" is used with the meaning "non-negative integer" and "fraction" is used with the meaning "non-negative rational number."

Another aspect of mathematics is the practice of defining concepts in terms of a small collection of fundamental concepts rather than treating concepts as unrelated. A small collection of fundamental concepts underlies the organization of the Standards. Definitions made in terms of these concepts become more explicit over the grades.• For example, subtraction can mean "take from," "find the unknown addend," or "find how much more (or less)," depending on context. However, as a mathematical operation subtraction can be defined in terms of the fundamental relation of addends and sum. Students acquire an informal understanding of this definition in Grade 1• and use it in solving problems throughout their mathematical work. The number line is another fundamental concept. In

• Note Standard for Mathematical Practice 6: "Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. . . . By the time they reach high school they have learned to examine claims and make explicit use of definitions."

• Note 1.OA.4: "Understand subtraction as an unknown-addend problem." Similarly, 3.OA.6: "Understand division as an unknown-factor problem."

elementary grades, students represent whole numbers (2.MD.6), then fractions (3.NF.2) on number line diagrams. Later, they understand integers and rational numbers (6.NS.6), then real numbers (8.NS.2), as points on the number line.•

• For further discussion, see "Overview of School Algebra" in U.S. Department of Education, 2008, "Report of the Task Group on Conceptual Knowledge and Skills" in Foundations for Success: The Final Report of the National Mathematics Advisory Panel.

Large-scale comparative studies One area of research compares aspects of educational systems in different countries. Compared to those of high-achieving countries, U.S. standards and curricula of recent decades were "a mile wide and an inch deep."•

• See Schmidt, Houang, & Cogan, 2002, "A Coherent Curriculum," *American Educator*, `http://aft.org/pdfs/americaneducator/summer2002/curriculum.pdf`.

In contrast, the organization of topics in high-achieving countries is more focused and more coherent. Focus refers to the number of topics taught at each grade and coherence is related to the way in which topics are organized. Curricula and standards that are focused have few topics in each grade. They are coherent if they are:

> articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential and hierarchical nature of the disciplinary content from which the subject matter derives.•

• Schmidt & Houang, 2012, "Curricular Coherence and the Common Core State Standards for Mathematics," *Educational Researcher*, `http://edr.sagepub.com/content/41/8/294`, p. 295.

Textbooks and curriculum documents from high-achieving countries give examples of such sequences of topics and performances.•

• For examples of "course of study" documents from other countries, see `http://bit.ly/eb601T`. Some textbooks from other countries are readily available. The University of Chicago School Mathematics Project (`http://bit.ly/18tEN7R`) has translations of Japanese textbooks for grades 7–9 and Russian grades 1–3. Singapore Math (`www.singaporemath.com`) has textbooks from Singapore. Global Education Resources (GER, `http://www.globaledresources.com`) has translations of Japanese textbooks for grades 1–6 and translations of the teaching guides for grades 1–6 and 7–9. Portions of the teachers manuals for the Japanese textbooks have been translated and can be downloaded at Lesson Study Group at Mills College (`http://bit.ly/12bZ1KQ`). The first page of a two-page diagram showing connections of topics for Grades 1–6 in Japan can be seen at `http://bit.ly/12EOjfN`.

Research on children's learning trajectories Within the United States, researchers who study children's learning have identified developmental sequences associated with constructs such as "teaching–learning paths," "learning progressions," or "learning trajectories." For example,

> A learning trajectory has three parts: a specific mathematical goal, a developmental path along which children develop to reach that goal, and a set of instructional activities that help children move along that path.•

• Clements & Sarama, 2009, *Learning and Teaching Early Math: The Learning Trajectories Approach*, Routledge, p. viii.

Findings from this line of research illuminate those of the large-scale comparative studies by giving details about how particular instructional activities help children develop specific mathematical abilities, identifying behavioral milestones along these paths.

The Progressions for the Common Core State Standards are not "learning progressions" in the sense described above. Well-documented learning progressions for all of K–12 mathematics do not exist. However, the Progressions for Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Geometry, and Geometric Measurement are informed by such learning progressions and are thus able to outline central instructional sequences and activities which have informed the Standards.•

• For more about research in this area, see the National Research Council's reports *Adding It Up: Helping Children Learn Mathematics*, 2001, and *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, 2009 (online at www.nap.edu); Sarama & Clements, 2009, *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children*, Routledge.

Other research on cognition and learning Other research on cognition, learning, and learning mathematics has informed the development of the Standards and Progressions in several ways. Fine-grained studies have identified cognitive features of learning and instruction for topics such as the equal sign in elementary and middle grades, proportional relationships, or connections among different representations of a linear function. Such studies have informed the development of standards in areas where learning progressions do not exist.• For example, it is possible for students in early grades to have a "relational" meaning for the equal sign, e.g., understanding $6 = 6$ and $7 = 8 - 1$ as correct equations (1.OA.7), rather than an "operational" meaning in which the right side of the equal sign is restricted to indicating the outcome of a computation. A relational understanding of the equal sign is associated with fewer obstacles in middle grades, and is consistent with its standard meaning in mathematics. Another example: Studies of students' interpretations of functions and graphs indicate specific features of desirable knowledge, e.g., that part of understanding is being able to identify and use the same properties of the same object in different representations. For instance, students identify the constant of proportionality (also known as the unit rate) in a graph, table, diagram, or equation of a proportional relationship (7.RP.2b) and can explain correspondences between its different representations (MP.1).

• For reports which summarize some research in these areas, see National Research Council, 2001, *Adding It Up: Helping Children Learn Mathematics*; National Council of Teachers of Mathematics, 2003, *A Research Companion to Principles and Standards for School Mathematics*; U.S. Department of Education, 2008, "Report of the Task Group on Learning Processes" in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. For recommendations that reflect research in these areas, see the National Council of Teachers of Mathematics reports *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, 2006 and *Focus in High School Mathematics: Reasoning and Sense Making*, 2009.

Studies in cognitive science have examined experts' knowledge, showing what the results of successful learning look like. Rather than being a collection of isolated facts, experts' knowledge is connected and organized according to underlying disciplinary principles.• So, for example, an expert's knowledge of multiplying whole numbers and mixed numbers, expanding binomials, and multiplying complex numbers is connected by common underlying principles rather than four separately memorized and unrelated special-purpose procedures. These findings from studies of experts are consistent with those of comparative research on curriculum. Both suggest that standards and curricula should attend to "key ideas that determine how knowledge is organized and generated within that discipline."•

• See the chapter on how experts differ from novices in the National Research Council's *How People Learn: Brain, Mind, Experience, and School* (online at `http://www.nap.edu/catalog.php?record_id=9853`).

• Schmidt & Houang, 2007, "Lack of Focus in the Intended Mathematics Curriculum: Symptom or Cause?" in *Lessons Learned: What International Assessments Tell Us About Math Achievement, Brookings Institution Press*.

The ways in which content knowledge is deployed (or not) are intertwined with mathematical dispositions and attitudes.• For example, in calculating 30×9, a third grade might use the simpler form of the original problem (MP.1): calculating $3 \times 9 = 27$, then multiplying the result by 10 to get 270 (3.NBT.3). Formulation of the Standards for Mathematical Practice drew on the process standards of the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics*, the strands of mathematical proficiency in the National Research Council's *Adding It Up*, and other distillations.•

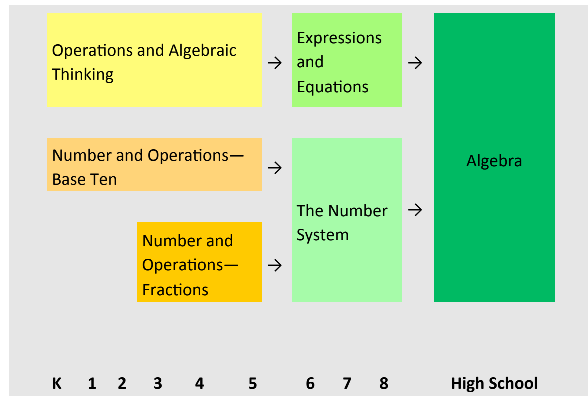
• See the discussions of self-monitoring, metacognition, and heuristics in How People Learn and the Problem Solving Standard of Principles and Standards for School Mathematics.

• See Harel, 2008, "What is Mathematics? A Pedagogical Answer to a Philosophical Question" in Gold & Simons (eds.), *Proof and Other Dilemmas*, Mathematical Association of America; Cuoco, Goldenberg, & Mark, 1996, "Habits of Mind: An Organizing Principle for a Mathematics Curriculum," *Journal of Mathematical Behavior* or Cuoco, 1998, "Mathematics as a Way of Thinking about Things," in *High School Mathematics at Work*, National Academies Press, which can be read online at `http://bit.ly/12Fa27m`. *Adding It Up* can be read online at `http://bit.ly/mbeQs1`.

Organization of the Common Core State Standards for Mathematics

An important feature of the Standards for Mathematical Content is their organization in groups of related standards. In K–8, these groups are called domains and in high school, they are called conceptual categories. The diagram below shows K–8 domains which are important precursors of the conceptual category of algebra.• In contrast, many standards and frameworks in the United States are presented as parallel K–12 "strands." Unlike the diagram in the margin, a strands type of presentation has the disadvantage of deemphasizing relationships of topics in different strands.

• For a more detailed diagram of relationships among the standards, see `http://commoncoretools.me/2012/06/09/jason-zimbas-wiring-diagram`.



Other aspects of the structure of the Standards are less obvious. The Progressions elaborate some features of this structure•, in particular:

• Because the Progressions focus on key ideas and the Standards have different levels of grain-size, not every standard is included in some Progression.

- Grade-level coordination of standards across domains.
- Connections between standards for content and for mathematical practice.
- Key ideas that develop within one domain over the grades.
- Key ideas that change domains as they develop over the grades.
- Key ideas that recur in different domains and conceptual categories.

Grade-level coordination of standards across domains or conceptual categories One example of how standards are coordinated is the following. In Grade 4 measurement and data, students solve problems involving conversion of measurements from a larger unit to a smaller unit.[4.MD.1] In Grade 5, this extends to conversion from smaller units to larger ones.[5.MD.1]

These standards are coordinated with the standards for operations on fractions. In Grade 4, expectations for multiplication are limited to multiplication of a fraction by a whole number (e.g., $3 \div 2/5$) and its representation by number line diagrams, other visual models, and equations. In Grade 5, fraction multiplication extends to multiplication of two non-whole number fractions.[5.NF.6, 5.MD.1]

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

5.MD.1 Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.MD.1 Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Connections between content and practice standards The Progressions provide examples of "points of intersection" between content and practice standards. For instance, standard algorithms for operations with multi-digit numbers can be viewed as expressions of regularity in repeated reasoning (MP.8). Such examples can be found by searching the Progressions electronically for "MP."

Key ideas within domains Within the domain of Number and Operations Base Ten, place value begins with the concept of ten ones in Kindergarten and extends through Grade 6, developing further in the context of whole number and decimal representations and computations.

Key ideas that change domains Some key concepts develop across domains and grades. For example, understanding number line diagrams begins in geometric measurement, then develops further in the context of fractions in Grade 3 and beyond.

Coordinated with the development of multiplication of fractions, measuring area begins in Grade 3 geometric measurement for rectangles with whole-number side lengths, extending to rectangles with fractional side lengths in Grade 5. Measuring volume begins in Grade 5 geometric measurement with right rectangular prisms with whole-number side lengths, extending to such prisms with fractional edge lengths in Grade 6 geometry.

Key recurrent ideas Among key ideas that occur in more than one domain or conceptual category are those of:

- composing and decomposing
- unit (including derived and subordinate unit).

These begin in elementary grades and continue through high school. Students develop tacit knowledge of these ideas by using them, which later becomes more explicit, particularly in algebra.

A group of objects can be decomposed without changing its cardinality, and this can be represented in equations. For example, a group of 4 objects can be decomposed into a group of 1 and a group of 3, and represented with various equations, e.g., $4 = 1+3$ or $1 + 3 = 4$. Properties of operations allow numerical expressions to be decomposed and rearranged without changing their value. For example, the 3 in $1 + 3$ can be decomposed as $1 + 2$ and, using the associative property, the expression can be rearranged as $2+2$. Variants of this idea (often expressed as "transforming" or "rewriting" an expression) occur throughout K–8, extending to algebra and other categories in high school.

One-, two-, and three-dimensional geometric figures can be decomposed and rearranged without changing—respectively—their length, area, or volume. For example, two copies of a square can be put edge-to-edge and be seen as composing a rectangle. A rectangle can be decomposed to form two triangles of the same shape. Variants of this idea (often expressed as "dissecting" and "rearranging") occur throughout K–8, extending to geometry and other categories in high school.

In K–8, an important occurrence of units is in the base-ten system for numbers. A whole number can be viewed as a collection of ones, or organized in terms of its base-ten units. Ten ones compose a unit called a ten. That unit can be decomposed as ten ones. Understanding place value involves understanding that each place of a base-ten numeral represents an amount of a base-ten unit: ones, tens, hundreds, ... and tenths, hundredths, etc. The regularity in composing and decomposing of base-ten units is a major feature used and highlighted by algorithms for computing operations on whole numbers and decimals.

Units occur as units of measurement for length, area, and volume in geometric measurement and geometry. Students iterate these units in measurement, first physically and later mentally, e.g., placing copies of a length unit side-by-side to measure length, tiling a region with copies of an area unit to measure area, or packing a container with copies of a volume unit to measure volume. They understand that a length unit determines derived units for area and volume, e.g., a meter gives rise to a square meter and cubic meter.

Representing amounts in terms of units

	two units	notation	one unit	notation
Base-ten units	1 ten, 3 ones	13	13 ones	–
Fractional units	1 one, 3 fifths	$1\frac{3}{5}$	8 fifths	$\frac{8}{5}$
Measurement units	1 foot, 3 inches	1 ft, 3 in	15 inches	15 in
Base-ten units	1 one, 3 tenths	1.3	13 tenths	–

An amount may be represented in terms of one unit or in terms of two units, where one unit is a composition of the other.

Students learn to decompose a one ("a whole") into subordinate units: unit fractions of equal size. The whole is a length (possibly represented by an endpoint) on the number line or is a single shape or object. When possible, students are able to write a number in terms of those units in different ways, as a fraction, decimal, or mixed number. They expand their conception of unit by learning to view a group of objects as a unit and partition that unit into unit fractions of equal size.

Students learn early that groups of objects or numbers can be decomposed and reassembled without changing their cardinality. Later, students learn that specific length, area, or volume units can be decomposed into subordinate units of equal size, e.g., a meter can be decomposed into decimeters, centimeters, or millimeters.

These ideas are extended in high school. For example, derived units may be created from two or more different units, e.g., miles per hour or vehicle-mile traveled. Shapes are decomposed and reassembled in order to determine certain attributes. For example, areas can be decomposed and reassembled as in the proof of the Pythagorean Theorem or angles can be decomposed and reassembled to yield trigonometric formulas.

Reconceptualized topics; changed notation and terminology

This section mentions some topics, terms, and notation that have been frequent in U.S. school mathematics, but do not occur in the Standards or Progressions.

"Number sentence" in elementary grades "Equation" is used instead of "number sentence," allowing the same term to be used throughout K–12.

Notation for remainders in division of whole numbers One aspect of attending to logical structure is attending to consistency. This has sometimes been neglected in U.S. school mathematics as illustrated by a common practice. The result of division within the system of whole numbers is frequently written like this:

$$84 \div 10 = 8 \text{ R } 4 \text{ and } 44 \div 5 = 8 \text{ R } 4.$$

Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation 8 R 4 does not indicate a number.

Rather than writing the result of division in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written like this:

$$84 = 8 \times 10 + 4 \text{ and } 44 = 8 \times 5 + 4.$$

Conversion and simplification To achieve the expectations of the Standards, students need to be able to transform and use numerical and symbolic expressions. The skills traditionally labeled "conversion" and "simplification" are a part of these expectations. As noted in the statement of Standard for Mathematical Practice 1, students transform a numerical or symbolic expression in order to get the information they need, using conversion, simplification, or other types of transformations. To understand correspondences between different approaches to the same problem or different representations for the same situation, students draw on their understanding of different representations for a given numerical or symbolic expression as well as their understanding of correspondences between equations, tables, graphs, diagrams, and verbal descriptions.

Fraction simplification, fraction-decimal-percent conversion In Grade 3, students recognize and generate equivalences between fractions in simple cases (3.NF.3). Two important building blocks for understanding relationships between fraction and decimal notation occur in Grades 4 and 5. In Grade 4, students' understanding of decimal notation for fractions includes using decimal notation for fractions with denominators 10 and 100 (4.NF.5; 4.NF.6). In Grade 5, students' understanding of fraction notation for decimals includes using fraction notation for decimals to thousandths (5.NBT.3a).

Students identify correspondences between different approaches to the same problem (MP.1). In Grade 4, when solving word problems that involve computations with simple fractions or decimals (e.g., 4.MD.2), one student might compute

$$\frac{1}{5} + \frac{12}{10}$$

as

$$.2 + 1.2 = 1.4,$$

another as

$$\frac{1}{5} + \frac{6}{5} = \frac{7}{5};$$

and yet another as

$$\frac{2}{10} + \frac{12}{10} = \frac{14}{10}.$$

Explanations of correspondences between

$$\frac{1}{5} + \frac{12}{10}, \quad .2 + 1.2, \quad \frac{1}{5} + \frac{6}{5}, \quad \text{and} \quad \frac{2}{10} + \frac{12}{10}$$

draw on understanding of equivalent fractions (3.NF.3 is one building block) and conversion from fractions to decimals (4.NF.5; 4.NF.6). This is revisited and augmented in Grade 7 when students use numerical and algebraic expressions to solve problems posed with rational numbers expressed in different forms, converting between forms as appropriate (7.EE.3).

In Grade 6, percents occur as rates per 100 in the context of finding parts of quantities (6.PR.3c). In Grade 7, students unify their understanding of numbers, viewing percents together with fractions and decimals as representations of rational numbers. Solving a wide variety of percent problems (7.RP.3) provides one source of opportunities to build this understanding.

Simplification of algebraic expressions In Grade 6, students apply properties of operations to generate equivalent expressions (6.EE.3). For example, they apply the distributive property to $3(2 + x)$ to generate $6 + 3x$. Traditionally, $6 + 3x$ is called the "simplification" of $3(2+x)$, however, students are not required to learn this terminology. Although the term "simplification" may suggest that the simplified form of an expression is always the most useful or always leads to a simpler form of a problem, this is not always the case. Thus, the use of this term may be misleading for students.

In Grade 7, students again apply properties of operations to generate equivalent expressions, this time to linear expressions with rational number coefficients (7.EE.1). Together with their understanding of fractions and decimals, students draw on their understanding of equivalent forms of an expression to identify and explain correspondences between different approaches to the same problem. For example, in Grade 7, this can occur in solving multi-step problems posed in terms of a mixture of fractions, decimals, and whole numbers (7.EE.4).

In high school, students apply properties of operations to solve problems, e.g., by choosing and producing an equivalent form of an expression for a quadratic or exponential function (A-SSE.3). As in earlier grades, the simplified form of an expression is one of its equivalent forms.

Terms and usage in the Standards and Progressions

In some cases, the Standards give choices or suggest a range of options. For example, standards like K.NBT.1, 4.NF.3c, and G-CO.12 give lists such as: "using objects or drawings"; "replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction"; "dynamic geometric software, compass and straightedge, reflective devices, and paper folding." Such lists are intended to suggest various possibilities rather than being comprehensive lists of requirements. The abbreviation "e.g." in a standard is frequently used as an indication that what follows is an example, not a specific requirement.

On the other hand, the Standards do impose some very important constraints. The structure of the Standards uses a particular definition of "fraction" for definitions and development of operations on fractions (see the Number and Operations—Fractions Progression). Likewise, the standards that concern ratio and rate rely on particular definitions of those terms. These are described in the Ratios and Proportional Relationships Progression.

Terms used in the Standards and Progressions are not intended as prescriptions for terms that teachers must use in the classroom. For example, students do not need to know the names of different types of addition situations, such as "put-together" or "compare," although these can be useful for classroom discourse. Likewise, Grade 2 students might use the term "line plot," its synonym "dot plot," or describe this type of diagram in some other way.

Progressions for the Common Core State Standards in Mathematics (draft)

29 May 2011

K, Counting and Cardinality; K–5, Operations and Algebraic Thinking

Counting and Cardinality and Operations and Algebraic Thinking are about understanding and using numbers. Counting and Cardinality underlies Operations and Algebraic Thinking as well as Number and Operations in Base Ten. It begins with early counting and telling how many in one group of objects. Addition, subtraction, multiplication, and division grow from these early roots. From its very beginnings, this Progression involves important ideas that are neither trivial nor obvious; these ideas need to be taught, in ways that are interesting and engaging to young students.

The Progression in Operations and Algebraic Thinking deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of the Progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measures, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers.

The generality of the concepts involved in Operations and Algebraic Thinking means that students' work in this area should be designed to help them extend arithmetic beyond whole numbers (see the NF and NBT Progressions) and understand and apply expressions and equations in later grades (see the EE Progression).

Addition and subtraction are the first operations studied. Initially, the meaning of addition is separate from the meaning of subtraction, and students build relationships between addition and subtraction over time. Subtraction comes to be understood as reversing the actions involved in addition and as finding an unknown addend. Likewise, the meaning of multiplication is initially separate from the meaning of division, and students gradually perceive relationships between division and multiplication analogous to those between addition and subtraction, understanding division as reversing the actions involved in multiplication and finding an unknown product.

Over time, students build their understanding of the properties of arithmetic: commutativity and associativity of addition and multiplication, and distributivity of multiplication over addition. Initially, they build intuitive understandings of these properties, and they use these intuitive understandings in strategies to solve real-world and mathematical problems. Later, these understandings become more explicit and allow students to extend operations into the system of rational numbers.

As the meanings and properties of operations develop, students develop computational methods in tandem. The OA Progression in Kindergarten and Grade 1 describes this development for single-digit addition and subtraction, culminating in methods that rely on properties of operations. The NBT Progression describes how these methods combine with place value reasoning to extend computation to multi-digit numbers. The NF Progression describes how the meanings of operations combine with fraction concepts to extend computation to fractions.

Students engage in the Standards for Mathematical Practice in grade-appropriate ways from Kindergarten to Grade 5. Pervasive classroom use of these mathematical practices in each grade affords students opportunities to develop understanding of operations and algebraic thinking.

Counting and Cardinality

Several progressions originate in knowing number names and the count sequence:[K.CC.1]

K.CC.1 Count to 100 by ones and by tens.

From saying the counting words to counting out objects Students usually know or can learn to say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects. Students become fluent in saying the count sequence so that they have enough attention to focus on the pairings involved in counting objects. To count a group of objects, they pair each word said with one object.[K.CC.4a] This is usually facilitated by an indicating act (such as pointing to objects or moving them) that keeps each word said in time paired to one and only one object located in space. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays (they need to ensure they reach every row or column and do not repeat rows or columns); circles (they need to stop just before the object they started with); and scattered configurations (they need to make a single path through all of the objects).[K.CC.5] Later, students can count out a given number of objects,[K.CC.5] which is more difficult than just counting that many objects, because counting must be fluent enough for the student to have enough attention to remember the number of objects that is being counted out.

K.CC.4a Understand the relationship between numbers and quantities; connect counting to cardinality.

a When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

From subitizing to single-digit arithmetic fluency Students come to quickly recognize the cardinalities of small groups without having to count the objects; this is called *perceptual subitizing*. Perceptual subitizing develops into *conceptual subitizing*—recognizing that a collection of objects is composed of two subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying "four"). Use of conceptual subitizing in adding and subtracting small numbers progresses to supporting steps of more advanced methods for adding, subtracting, multiplying, and dividing single-digit numbers (in several OA standards from Grade 1 to 3 that culminate in single-digit fluency).

From counting to counting on Students understand that the last number name said in counting tells the number of objects counted.[K.CC.4b] Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the

K.CC.4b Understand the relationship between numbers and quantities; connect counting to cardinality.

b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

more advanced counting-on methods in which a counting word represents a group of objects that are added or subtracted and addends become embedded within the total[1.OA.6] (see page 14). Being able to count forward, beginning from a given number within the known sequence,[K.CC.2] is a prerequisite for such counting on. Finally, understanding that each successive number name refers to a quantity that is one larger[K.CC.4c] is the conceptual start for Grade 1 counting on. Prior to reaching this understanding, a student might have to recount entirely a collection of known cardinality to which a single object has been added.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

K.CC.2 Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

From spoken number words to written base-ten numerals to base-ten system understanding The NBT Progression discusses the special role of 10 and the difficulties that English speakers face because the base-ten structure is not evident in all the English number words.

From comparison by matching to comparison by numbers to comparison involving adding and subtracting The standards about comparing numbers[K.CC.6,K.CC.7] focus on students identifying which of two groups has more than (or fewer than, or the same amount as) the other. Students first learn to match the objects in the two groups to see if there are any extra and then to count the objects in each group and use their knowledge of the count sequence to decide which number is greater than the other (the number farther along in the count sequence). Students learn that even if one group looks as if it has more objects (e.g., has some extra sticking out), matching or counting may reveal a different result. Comparing numbers progresses in Grade 1 to adding and subtracting in comparing situations (finding out "how many more" or "how many less"[1.OA.1] and not just "which is more" or "which is less").

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Operations and Algebraic Thinking

Overview of Grades K–2

Students develop meanings for addition and subtraction as they encounter problem situations in Kindergarten, and they extend these meanings as they encounter increasingly difficult problem situations in Grade 1. They represent these problems in increasingly sophisticated ways. And they learn and use increasingly sophisticated computation methods to find answers. In each grade, the situations, representations, and methods are calibrated to be coherent and to foster growth from one grade to the next.

The main addition and subtraction situations students work with are listed in Table 1. The computation methods they learn to use are summarized in the margin and described in more detail in the Appendix.

Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away. Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2. Counting On. Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Level 3. Convert to an Easier Problem. Decompose an addend and compose a part with another addend.

See Appendix for examples and further details.

Table 1: Addition and subtraction situations

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A+B=\square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A+\square=C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square+B=C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C-B=\square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C-\square=A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square-B=A$
	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A+B=\square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C=\square+\square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A+\square=C$ $C-A=\square$
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A+\square=C$ $C-A=\square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A+B=\square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C-B=\square$ $\square+B=C$

In each type (shown as a row), any one of the three quantities in the situation can be unknown, leading to the subtypes shown in each cell of the table. The table also shows some important language variants which, while mathematically the same, require separate attention. Other descriptions of the situations may use somewhat different names. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

Kindergarten

Students act out adding and subtracting situations by representing quantities in the situation with objects, their fingers, and math drawings (MP5).[K.OA.1] To do this, students must mathematize a real-world situation (MP4), focusing on the quantities and their relationships rather than non-mathematical aspects of the situation. Situations can be acted out and/or presented with pictures or words. Math drawings facilitate reflection and discussion because they remain after the problem is solved. These concrete methods that show all of the objects are called Level 1 methods.

K.OA.1 Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

Students learn and use mathematical and non-mathematical language, especially when they make up problems and explain their representation and solution. The teacher can write expressions (e.g., $3-1$) to represent operations, as well as writing equations that represent the whole situation before the solution (e.g., $3-1=\square$) or after (e.g., $3-1=2$). Expressions like $3-1$ or $2+1$ show the operation, and it is helpful for students to have experience just with the expression so they can conceptually chunk this part of an equation.

Working within 5 Students work with small numbers first, though many kindergarteners will enter school having learned parts of the Kindergarten standards at home or at a preschool program. Focusing attention on small groups in adding and subtracting situations can help students move from perceptual subitizing to conceptual subitizing in which they see and say the addends and the total,• e.g., "Two and one make three."

• Note on vocabulary: The term "total" is used here instead of the term "sum." "Sum" sounds the same as "some," but has the opposite meaning. "Some" is used to describe problem situations with one or both addends unknown, so it is better in the earlier grades to use "total" rather than "sum." Formal vocabulary for subtraction ("minuend" and "subtrahend") is not needed for Kindergarten, Grade 1, and Grade 2, and may inhibit students seeing and discussing relationships between addition and subtraction. At these grades, the terms "total" and "addend" are sufficient for classroom discussion.

Students will generally use fingers for keeping track of addends and parts of addends for the Level 2 and 3 methods used in later grades, so it is important that students in Kindergarten develop rapid visual and kinesthetic recognition of numbers to 5 on their fingers. Students may bring from home different ways to show numbers with their fingers and to raise (or lower) them when counting. The three major ways used around the world are starting with the thumb, the little finger, or the pointing finger (ending with the thumb in the latter two cases). Each way has advantages physically or mathematically, so students can use whatever is familiar to them. The teacher can use the range of methods present in the classroom, and these methods can be compared by students to expand their understanding of numbers. Using fingers is not a concern unless it remains at the first level of direct modeling in later grades.

Students in Kindergarten work with the following types of addition and subtraction situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown (see the dark shaded types in Table 2). Add To/Take From situations are action-oriented; they show changes from an initial state to a final state. These situations are readily modeled by equations because each aspect of the situation has a representation as number, operation (+ or −), or equal sign (=, here with the meaning of "becomes," rather than the more general "equals").

Table 2: Addition and subtraction situations by grade level.

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$
	Total Unknown	**Both Addends Unknown[1]**	**Addend Unknown[2]**
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

Darker shading indicates the four Kindergarten problem subtypes. Grade 1 and 2 students work with all subtypes and variants. Unshaded (white) problems are the four difficult subtypes or variants that students should work with in Grade 1 but need not master until Grade 2. Adapted from CCSS, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

In Put Together/Take Apart situations, two quantities jointly compose a third quantity (the total), or a quantity can be decomposed into two quantities (the addends). This composition/decomposition may be physical or conceptual. These situations are acted out with objects initially and later children begin to move to conceptual mental actions of shifting between seeing the addends and seeing the total (e.g., seeing children or seeing boys and girls, or seeing red and green apples or all the apples).

The relationship between addition and subtraction in the Add To/Take From and the Put Together/Take Apart action situations is that of reversibility of actions: an Add To situation undoes a Take From situation and vice versa and a composition (Put Together) undoes a decomposition (Take Apart) and vice versa.

Put Together/Take Apart situations with Both Addends Unknown play an important role in Kindergarten because they allow students to explore various compositions that make each number.[K.OA.3] This will help students to build the Level 2 embedded number representations used to solve more advanced problem subtypes. As students decompose a given number to find all of the partners• that compose the number, the teacher can record each decomposition with an equation such as $5 = 4 + 1$, showing the total on the left and the two addends on the right.• Students can find patterns in all of the decompositions of a given number and eventually summarize these patterns for several numbers.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4 + 1$).

• The two addends that make a total can also be called partners in Kindergarten and Grade 1 to help children understand that they are the two numbers that go together to make the total.

• For each total, two equations involving 0 can be written, e.g., $5 = 5 + 0$ and $5 = 0 + 5$. Once students are aware that such equations can be written, practice in decomposing is best done without such 0 cases.

Equations with one number on the left and an operation on the right (e.g., $5 = 2 + 3$ to record a group of 5 things decomposed as a group of 2 things and a group of 3 things) allow students to understand equations as showing in various ways that the quantities on both sides have the same value.[MP6]

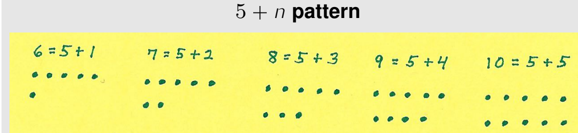
MP6 Working toward "using the equal sign consistently and appropriately."

Working within 10 Students expand their work in addition and subtraction from within 5 to within 10. They use the Level 1 methods developed for smaller totals as they represent and solve problems with objects, their fingers, and math drawings. Patterns such as "adding one is just the next counting word"[K.CC.4c] and "adding zero gives the same number" become more visible and useful for all of the numbers from 1 to 9. Patterns such as the $5 + n$ pattern used widely around the world play an important role in learning particular additions and subtractions, and later as patterns in steps in the Level 2 and 3 methods. Fingers can be used to show the same 5-patterns, but students should be asked to explain these relationships explicitly because they may not be obvious to all students.[MP3] As the school year progresses, students internalize their external representations and solution actions, and mental images become important in problem representation and solution.

K.CC.4c Understand the relationship between numbers and quantities; connect counting to cardinality.

c Understand that each successive number name refers to a quantity that is one larger.

$5 + n$ **pattern**



MP3 Students explain their conclusions to others.

Student drawings show the relationships in addition and subtraction situations for larger numbers (6 to 9) in various ways, such as groupings, things crossed out, numbers labeling parts or totals, and letters or words labeling aspects of the situation. The symbols $+$, $-$, or $=$ may be in the drawing. Students should be encouraged to explain their drawings and discuss how different drawings are the same and different.[MP1]

Later in the year, students solve addition and subtraction equations for numbers within 5, for example, $2+1=\square$ or $3-1=\square$, while still connecting these equations to situations verbally or with drawings. Experience with decompositions of numbers and with Add To and Take From situations enables students to begin to fluently add and subtract within 5.[K.OA.5]

Finally, composing and decomposing numbers from 11 to 19 into ten ones and some further ones builds from all this work.[K.NBT.1] This is a vital first step kindergarteners must take toward understanding base-ten notation for numbers greater than 9. (See the NBT Progression.)

The Kindergarten standards can be stated succinctly, but they represent a great deal of focused and rich interactions in the classroom. This is necessary in order to enable all students to understand all of the numbers and concepts involved. Students who enter Kindergarten without knowledge of small numbers or of counting to ten will require extra teaching time in Kindergarten to meet the standards. Such time and support are vital for enabling all students to master the Grade 1 standards in Grade 1.

MP1 Understand the approaches of others and identify correspondences

K.OA.5 Fluently add and subtract within 5.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Grade 1

Students extend their work in three major and interrelated ways, by:

- Representing and solving a new type of problem situation (Compare);
- Representing and solving the subtypes for all unknowns in all three types;
- Using Level 2 and Level 3 methods to extend addition and subtraction problem solving beyond 10, to problems within 20. In particular, the OA progression in Grade 1 deals with adding two single-digit addends, and related subtractions.•

• Other Grade 1 problems within 20, such as $14 + 5$, are best viewed in the context of place value, i.e., associated with 1.NBT.4. See the NBT Progression.

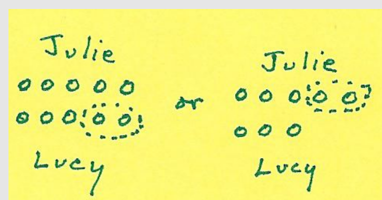
Representing and solving a new type of problem situation (Compare) In a Compare situation, two quantities are compared to find "How many more" or "How many less."•[K.CC.6,K.CC.7] One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown.

• Compare problems build upon Kindergarten comparisons, in which students identified "Which is more?" or "Which is less?" without ascertaining the difference between the numbers.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

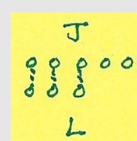
K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

Representing the difference in a Compare problem

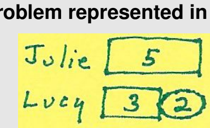


The language of comparisons is also difficult. For example, "Julie has three more apples than Lucy" tells both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form. Another language issue is that the comparing sentence might be stated in either of two related ways, using "more" or "less." Students need considerable experience with "less" to differentiate it from "more"; some children think that "less" means "more." Finally, as well as the basic "How many more/less" question form, the comparing sentence might take an active, equalizing and counterfactual form (e.g., "How many more apples does Lucy need to have as many as Julie?") or might be stated in a static and factual way as a question about how many things are unmatched (e.g., "If there are 8 trucks and 5 drivers, how many trucks do not have a driver?"). Extensive experience with a variety of contexts is needed to master these linguistic and situational complexities. Matching with objects and with drawings, and labeling each quantity (e.g., J or Julie and L or Lucy) is helpful. Later in Grade 1, a tape diagram can be used. These comparing diagrams can continue to be used for multi-digit numbers, fractions, decimals, and variables, thus connecting understandings of these numbers in

Compare problem solved by matching



Compare problem represented in tape diagram



comparing situations with such situations for single-digit numbers. The labels can get more detailed in later grades.

Some textbooks represent all Compare problems with a subtraction equation, but that is not how many students think of the subtypes. Students represent Compare situations in different ways, often as an unknown addend problem (see Table 1). If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context.

Representing and solving the subtypes for all unknowns in all three types In Grade 1, students solve problems of all twelve subtypes (see Table 2) including both language variants of Compare problems. Initially, the numbers in such problems are small enough that students can make math drawings showing all the objects in order to solve the problem. Students then represent problems with equations, called situation equations. For example, a situation equation for a Take From problem with Result Unknown might read $14 - 8 = \square$.

Put Together/Take Apart problems with Addend Unknown afford students the opportunity to see subtraction as the opposite of addition in a different way than as reversing the action, namely as finding an unknown addend.[1.OA.4] The meaning of subtraction as an unknown-addend addition problem is one of the essential understandings students will need in middle school in order to extend arithmetic to negative rational numbers.

[1.OA.4] Understand subtraction as an unknown-addend problem.

Students next gain experience with the more difficult and more "algebraic" problem subtypes in which a situation equation does not immediately lead to the answer. For example, a student analyzing a Take From problem with Change Unknown might write the situation equation $14 - \square = 8$. This equation does not immediately lead to the answer. To make progress, the student can write a related equation called a solution equation—in this case, either $8 + \square = 14$ or $14 - 8 = \square$. These equations both lead to the answer by Level 2 or Level 3 strategies (see discussion in the next section).

Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. Learning where the total is in addition equations (alone on one side of the equal sign) and in subtraction equations (to the left of the minus sign) helps students move from a situation equation to a related solution equation.

Because the language and conceptual demands are high, some students in Grade 1 may not master the most difficult subtypes of word problems, such as Compare problems that use language opposite to the operation required for solving (see the unshaded subtypes and variants in Table 2). Some students may also still have difficulty with the conceptual demands of Start Unknown problems. Grade 1 children should have an opportunity to solve and discuss such problems, but proficiency on grade level tests with these most difficult subtypes should wait until Grade 2 along with the other extensions of problem solving.

Using Level 2 and Level 3 strategies to extend addition and subtraction problem solving beyond 10, to problems within 20 As Grade 1 students are extending the range of problem types and subtypes they can solve, they are also extending the range of numbers they deal with[1.OA.6] and the sophistication of the methods they use to add and subtract within this larger range.[1.OA.1,1.OA.8]

The advance from Level 1 methods to Level 2 methods can be clearly seen in the context of situations with unknown addends.[1] These are the situations that can be represented by an addition equation with one unknown addend, e.g., $9 + \square = 13$. Students can solve some unknown addend problems by trial and error or by knowing the relevant decomposition of the total. But a Level 2 counting on solution involves seeing the 9 as part of 13, and understanding that counting the 9 things can be "taken as done" if we begin the count from 9: thus the student may say,

"Niiiiine,	ten,	eleven,	twelve,	thirteen."
	1	2	3	4

Students keep track of how many they counted on (here, 4) with fingers, mental images, or physical actions such as head bobs. Elongating the first counting word ("Niiiiine...") is natural and indicates that the student differentiates between the first addend and the counts for the second addend. Counting on enables students to add and subtract easily within 20 because they do not have to use fingers to show totals of more than 10 which is difficult. Students might also use the commutative property to shorten tasks, by counting on from the larger addend even if it is second (e.g., for $4 + 9$, counting on from 9 instead of from 4).

Counting on should be seen as a thinking strategy, not a rote method. It involves seeing the first addend as embedded in the total, and it involves a conceptual interplay between counting and the cardinality in the first addend (shifting from the cardinal meaning of the first addend to the counting meaning). Finally, there is a level of abstraction involved in counting on, because students are counting

[1]Grade 1 students also solve the easy Kindergarten problem subtypes by counting on.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

1.OA.8 Determine the unknown whole number in an addition or subtraction equation relating three whole numbers.

the words rather than objects. Number words have become objects to students.

Counting on can be used to add (find a total) or subtract (find an unknown addend). To an observer watching the student, adding and subtracting look the same. Whether the problem is $9+4$ or $13-9$, we will hear the student say the same thing: "Niiiiine, ten, eleven, twelve, thirteen" with four head bobs or four fingers unfolding. The differences are in what is being monitored to know when to stop, and what gives the answer.

Counting on to add and subtract

$9+4$

"Niiiine, ten, eleven, twelve, thirteen."

1 2 3 4

$13-9$

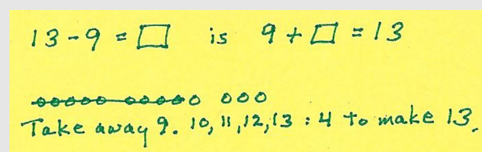
"Niiiine, ten, eleven, twelve, thirteen."

1 2 3 4

When counting on to add $9+4$, the student is counting the fingers or head bobs to know when to stop counting aloud, and the last counting word said gives the answer. For counting on to subtract $13-9$, the opposite is true: the student is listening to counting words to know when to stop, and the accumulated fingers or head bobs give the answer.

Students in many countries learn counting forward methods of subtracting, including counting on. Counting on for subtraction is easier than counting down. Also, unlike counting down, counting on reinforces that subtraction is an unknown-addend problem. Learning to think of and solve subtractions as unknown addend problems makes subtraction as easy as addition (or even easier), and it emphasizes the relationship between addition and subtraction. The taking away meaning of subtraction can be emphasized within counting on by showing the total and then taking away the objects that are at the *beginning*. In a drawing this taking away can be shown with a horizontal line segment suggesting a minus sign. So one can think of the $9+\square=13$ situation as "I took away 9. I now have 10, 11, 12, 13 [stop when I hear 13], so 4 are left because I counted on 4 from 9 to get to 13." Taking away objects at the end suggests counting down, which is more difficult than counting on. Showing 13 decomposed in groups of five as in the illustration to the right also supports students seeing how to use the Level 3 make-a-ten method; 9 needs 1 more to make 10 and there are 3 more in 13, so 4 from 9 to 13.

"Taking away" indicated with horizontal line segment and solving by counting on to 13



Level 3 methods involve decomposing an addend and composing it with the other addend to form an equivalent but easier problem. This relies on properties of operations.1.OA.3 Students do not necessarily have to justify their representations or solution using properties, but they can begin to learn to recognize these properties in action and discuss their use after solving.

1.OA.3 Apply properties of operations as strategies to add and subtract.

There are a variety of methods to change to an easier problem. These draw on addition of three whole numbers.1.OA.2 A known addition or subtraction can be used to solve a related addition or subtraction by decomposing one addend and composing it with the other addend. For example, a student can change $8+6$ to the easier $10+4$ by decomposing $6=2+4$ and composing the 2 with the 8 to make 10: $8+6=8+2+4=10+4=14$.

1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

This method can also be used to subtract by finding an unknown addend: $14-8=\square$, so $8+\square=14$, so $14=8+2+4=8+6$, that is $14-8=6$. Students can think as for adding above (stopping when they reach 14), or they can think of taking 8 from 10, leaving 2 with the 4, which makes 6. One can also decompose with respect to ten: $13-4=13-3-1=10-1=9$, but this can be more difficult than the forward methods.

These make-a-ten methods• have three prerequisites reaching

• Computing 8 + 6 by making a ten

a. 8's partner to 10 is 2, so decompose 6 as 2 and its partner.

b. 2's partner to 6 is 4.

c. 10 + 4 is 14.

back to Kindergarten:

a. knowing the partner that makes 10 for any number (K.OA.4 sets the stage for this),

b. knowing all decompositions for any number below 10 (K.OA.3 sets the stage for this), and

c. knowing all teen numbers as $10+n$ (e.g., $12 = 10+2$, $15 = 10+5$, see K.NBT.1 and 1.NBT.2b).

The make-a-ten methods are more difficult in English than in East Asian languages in which teen numbers are spoken as *ten, ten one, ten two, ten three*, etc. In particular, prerequisite c is harder in English because of the irregularities and reversals in the teen number words.•

Another Level 3 method that works for certain numbers is a doubles ± 1 or ± 2 method: $6+7 = 6+(6+1) = (6+6)+1 = 12+1 = 13$. These methods do not connect with place value the way make-a-ten methods do.

The Add To and Take From Start Unknown situations are particularly challenging with the larger numbers students encounter in Grade 1. The situation equation $\square + 6 = 15$ or $\square - 6 = 9$ can be rewritten to provide a solution. Students might use the commutative property of addition to change $\square + 6 = 15$ to $6 + \square = 15$, then count on or use Level 3 methods to compose 4 (to make ten) plus 5 (ones in the 15) to find 9. Students might reverse the action in the situation represented by $\square - 6 = 9$ so that it becomes $9+6 = \square$. Or they might use their knowledge that the total is the first number in a subtraction equation and the last number in an addition equation to rewrite the situation equation as a solution equation: $\square - 6 = 9$ becomes $9+6 = \square$ or $6+9 = \square$.

The difficulty levels in Compare problems differ from those in Put Together/Take Apart and Add To and Take From problems. Difficulties arise from the language issues mentioned before and especially from the opposite language variants where the comparing sentence suggests an operation opposite to that needed for the solution.•

As students progress to Level 2 and Level 3 methods, they no longer need representations that show each quantity as a group of objects. Students now move on to diagrams that use numbers and show relationships between these numbers. These can be extensions of drawings made earlier that did show each quantity as a group of objects. Add To/Take From situations at this point can continue to be represented by equations. Put Together/Take Apart situations can be represented by the example drawings shown in the margin. Compare situations can be represented with tape diagrams showing the compared quantities (one smaller and one larger) and the difference. Other diagrams showing two numbers and the unknown can also be used. Such diagrams are a major step forward because the same diagrams can represent the adding and subtracting situations

K.OA.4 For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4+1$).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

1.NBT.2b Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

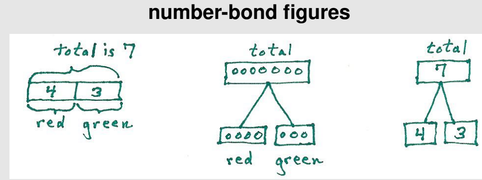
b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

• For example, "four" is spoken first in "fourteen," but this order is reversed in the numeral 14.

• Bigger Unknown: *"Fewer" version suggests wrong operation.* Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have?

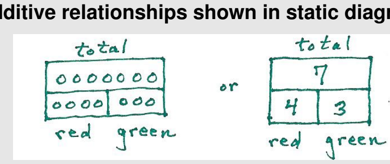
Smaller Unknown. *"More" version suggests wrong operation.* Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have?

Additive relationship shown in tape, part-whole, and number-bond figures



The tape diagram shows the addends as the tapes and the total (indicated by a bracket) as a composition of those tapes. The part-whole diagram and number-bond diagram capture the composing-decomposing action to allow the representation of the total at the top and the addends at the bottom either as drawn quantities or as numbers.

Additive relationships shown in static diagrams



Students sometimes have trouble with static part-whole diagrams because these display a double representation of the total and the addends (the total 7 above and the addends 4 and 3 below), but at a given time in the addition or subtraction situation not all three quantities are present. The action of moving from the total to the addends (or from the addends to the total) in the number-bond diagram reduces this conceptual difficulty.

for all of the kinds of numbers students encounter in later grades (multi-digit whole numbers, fractions, decimals, variables). Students can also continue to represent any situation with a situation equation and connect such equations to diagrams.[MP1] Such connections can help students to solve the more difficult problem situation subtypes by understanding where the totals and addends are in the equation and rewriting the equation as needed.

[MP1] By relating equations and diagrams, students work toward this aspect of MP1: Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs.

Grade 2

Grade 2 students build upon their work in Grade 1 in two major ways.[2.OA.1] They represent and solve situational problems of all three types which involve addition and subtraction within 100 rather than within 20, and they represent and solve two-step situational problems of all three types.

Diagrams used in Grade 1 to show how quantities in the situation are related continue to be useful in Grade 2, and students continue to relate the diagrams to situation equations. Such relating helps students rewrite a situation equation like $\square - 38 = 49$ as $49 + 38 = \square$ because they see that the first number in the subtraction equation is the total. Each addition and subtraction equation has seven related equations. Students can write all of these equations, continuing to connect addition and subtraction, and their experience with equations of various forms.

Because there are so many problem situation subtypes, there are many possible ways to combine such subtypes to devise two-step problems. Because some Grade 2 students are still developing proficiency with the most difficult subtypes, two-step problems should not involve these subtypes. Most work with two-step problems should involve single-digit addends.

Most two-step problems made from two easy subtypes are easy to represent with an equation, as shown in the first two examples to the right. But problems involving a comparison or two middle difficulty subtypes may be difficult to represent with a single equation and may be better represented by successive drawings or some combination of a diagram for one step and an equation for the other (see the last three examples). Students can make up any kinds of two-step problems and share them for solving.

The deep extended experiences students have with addition and subtraction in Kindergarten and Grade 1 culminate in Grade 2 with students becoming fluent in single-digit additions and the related subtractions using the mental Level 2 and 3 strategies as needed.[2.OA.2] So fluency in adding and subtracting single-digit numbers has progressed from numbers within 5 in Kindergarten to within 10 in Grade 1 to within 20 in Grade 2. The methods have also become more advanced.

The word *fluent* is used in the Standards to mean “fast and accurate.” Fluency in each grade involves a mixture of just knowing some answers, knowing some answers from patterns (e.g., “adding 0 yields the same number”), and knowing some answers from the use of strategies. It is important to push sensitively and encouragingly toward fluency of the designated numbers at each grade level, recognizing that fluency will be a mixture of these kinds of thinking which may differ across students. The extensive work relating addition and subtraction means that subtraction can frequently be solved by thinking of the related addition, especially for smaller numbers. It is also important that these patterns, strategies and decomposi-

[2.OA.1] Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Related addition and subtraction equations

$87-38=49$	$87-49=38$	$38+49=87$	$49+38=87$
$49=87-38$	$38=87-49$	$87=38+49$	$87=49+38$

Examples of two-step Grade 2 word problems

Two easy subtypes with the same operation, resulting in problems represented as, for example, $9+5+7=\square$ or $16-8-5=\square$ and perhaps by drawings showing these steps:

> Example for $9+5+7$: There were 9 blue balls and 5 red balls in the bag. Aki put in 7 more balls. How many balls are in the bag altogether?

Two easy subtypes with opposite operations, resulting in problems represented as, for example, $9-5+7=\square$ or $16+8-5=\square$ and perhaps by drawings showing these steps:

> Example for $9-5+7$: There were 9 carrots on the plate. The girls ate 5 carrots. Mother put 7 more carrots on the plate. How many carrots are there now?

One easy and one middle difficulty subtype:

> For example: Maria has 9 apples. Corey has 4 fewer apples than Maria. How many apples do they have in all?

> For example: The zoo had 7 cows and some horses in the big pen. There were 15 animals in the big pen. Then 4 more horses ran into the big pen. How many horses are there now?

Two middle difficulty subtypes:

> For example: There were 9 boys and some girls in the park. In all, 15 children were in the park. Then some more girls came. Now there are 14 girls in the park. How many more girls came to the park?

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

tions still be available in Grade 3 for use in multiplying and dividing and in distinguishing adding and subtracting from multiplying and dividing. So the important press toward fluency should also allow students to fall back on earlier strategies when needed. By the end of the K–2 grade span, students have sufficient experience with addition and subtraction to know single-digit sums from memory;[2.OA.2] as should be clear from the foregoing, this is not a matter of instilling facts divorced from their meanings, but rather as an outcome of a multi-year process that heavily involves the interplay of practice and reasoning.

Extensions to other standard domains and to higher grades In Grades 2 and 3, students continue and extend their work with adding and subtracting situations to length situations[2.MD.5,2.MD.6] (addition and subtraction of lengths is part of the transition from whole number addition and subtraction to fraction addition and subtraction) and to bar graphs.[2.MD.10,3.MD.3] Students solve two-step[3.OA.8] and multistep[4.OA.3] problems involving all four operations. In Grades 3, 4, and 5, students extend their understandings of addition and subtraction problem types in Table 1 to situations that involve fractions and decimals. Importantly, the situational meanings for addition and subtraction remain the same for fractions and decimals as for whole numbers.

[2.OA.2]Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

[2.MD.5]Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[2.MD.6]Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.10]Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

[3.MD.3]Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step “how many more” and “how many less” problems using information presented in scaled bar graphs.

[3.OA.8]Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

[4.OA.3]Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Summary of K–2 Operations and Algebraic Thinking

Kindergarten Students in Kindergarten work with three kinds of problem situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown. The numbers in these problems involve addition and subtraction within 10. Students represent these problems with concrete objects and drawings, and they find the answers by counting (Level 1 method). More specifically,

- For Add To with Result Unknown, they make or draw the starting set of objects and the change set of objects, and then they count the total set of objects to give the answer.
- For Take From with Result Unknown, they make or draw the starting set and "take away" the change set; then they count the remaining objects to give the answer.
- For Put Together/Take Apart with Total Unknown, they make or draw the two addend sets, and then they count the total number of objects to give the answer.

Grade 1 Students in Grade 1 work with all of the problem situations, including all subtypes and language variants. The numbers in these problems involve additions involving single-digit addends, and the related subtractions. Students represent these problems with math drawings and with equations.

Students master the majority of the problem types. They might sometimes use trial and error to find the answer, or they might just know the answer based on previous experience with the given numbers. But as a general method they learn how to find answers to these problems by counting on (a Level 2 method), and they understand and use this method.[1.OA.5,1.OA.6] Students also work with Level 3 methods that change a problem to an easier equivalent problem.[1.OA.3,1.OA.6] The most important of these Level 3 methods involve making a ten, because these methods connect with the place value concepts students are learning in this grade (see the NBT Progression) and work for any numbers. Students also solve the easier problem subtypes with these Level 3 methods.

The four problem subtypes that Grade 1 students should work with, but need not master, are:

- Add To with Start Unknown
- Take From with Start Unknown
- Compare with Bigger Unknown using "fewer" language (misleading language suggesting the wrong operation)
- Compare with Smaller Unknown using "more" language (misleading language suggesting the wrong operation)

1.OA.5 Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

1.OA.3 Apply properties of operations as strategies to add and subtract.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

Grade 2 Students in Grade 2 master all of the problem situations and all of their subtypes and language variants. The numbers in these problems involve addition and subtraction within 100. They represent these problems with diagrams and/or equations. For problems involving addition and subtraction within 20, more students master Level 3 methods; increasingly for addition problems, students might just know the answer (by end of Grade 2, students know all sums of two-digit numbers from memory[2.OA.2]). For other problems involving numbers to 100, Grade 2 students use their developing place value skills and understandings to find the answer (see the NBT Progression). Students work with two-step problems, especially with single-digit addends, but do not work with two-step problems in which both steps involve the most difficult problem subtypes and variants.

[2.OA.2] Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.

Grade 3

Students focus on understanding the meaning and properties of multiplication and division and on finding products of single-digit multiplying and related quotients.[3.OA.1–7] These skills and understandings are crucial; students will rely on them for years to come as they learn to multiply and divide with multi-digit whole number and to add, subtract, multiply and divide with fractions and with decimals. Note that mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding,[3.OA.7] may be quite time consuming because there are no general strategies for multiplying or dividing all single-digit numbers as there are for addition and subtraction. Instead, there are many patterns and strategies dependent upon specific numbers. So it is imperative that extra time and support be provided if needed.

Common types of multiplication and division situations. Common multiplication and division situations are shown in Table 3. There are three major types, shown as rows of Table 3. The Grade 3 standards focus on Equal Groups and on Arrays.• As with addition and subtraction, each multiplication or division situation involves three quantities, each of which can be the unknown. Because there are two factors and one product in each situation (product = factor × factor), each type has one subtype solved by multiplication (Unknown Product) and two unknown factor subtypes solved by division.

3.OA.1 Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

3.OA.2 Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each.

3.OA.3 Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

3.OA.4 Determine the unknown whole number in a multiplication or division equation relating three whole numbers.

3.OA.5 Apply properties of operations as strategies to multiply and divide.

3.OA.6 Understand division as an unknown-factor problem.

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

• Multiplicative Compare situations are more complex than Equal Groups and Arrays, and must be carefully distinguished from additive Compare problems. Multiplicative comparison first enters the Standards at Grade 4.[4.OA.1] For more information on multiplicative Compare problems, see the Grade 4 section of this progression.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

Table 3: Multiplication and division situations

	$A \times B = \square$	$A \times \square = C$ and $C \div A = \square$	$\square \times B = C$ and $C \div B = \square$
Equal Groups of Objects	Unknown Product There are A bags with B plums in each bag. How many plums are there in all?	Group Size Unknown If C plums are shared equally into A bags, then how many plums will be in each bag?	Number of Groups Unknown If C plums are to be packed B to a bag, then how many bags are needed?
Arrays of Objects		*Equal groups language*	
	Unknown Product There are A rows of apples with B apples in each row. How many apples are there?	Unknown Factor If C apples are arranged into A equal rows, how many apples will be in each row?	Unknown Factor If C apples are arranged into equal rows of B apples, how many rows will there be?
		Row and column language	
	Unknown Product The apples in the grocery window are in A rows and B columns. How many apples are there?	Unknown Factor If C apples are arranged into an array with A rows, how many columns of apples are there?	Unknown Factor If C apples are arranged into an array with B columns, how many rows are there?
Compare		$A > 1$	
	Larger Unknown A blue hat costs \$$B$. A red hat costs A times as much as the blue hat. How much does the red hat cost?	Smaller Unknown A red hat costs \$$C$ and that is A times as much as a blue hat costs. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. How many times as much does the red hat cost as the blue hat?
		$A < 1$	
	Smaller Unknown A blue hat costs \$$B$. A red hat costs A as much as the blue hat. How much does the red hat cost?	Larger Unknown A red hat costs \$$C$ and that is A of the cost of a blue hat. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. What fraction of the cost of the blue hat is the cost of the red hat?

Adapted from box 2–4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33.

Notes

Equal groups problems can also be stated in terms of columns, exchanging the order of A and B, so that the same array is described. For example: There are B columns of apples with A apples in each column. How many apples are there?

In the row and column situations (as with their area analogues), number of groups and group size are not distinguished.

Multiplicative Compare problems appear first in Grade 4, with whole-number values for A, B, and C, and with the "times as much" language in the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit fraction language change the subject of the comparing sentence, e.g., "A red hat costs A times as much as the blue hat" results in the same comparison as "A blue hat costs $1/A$ times as much as the red hat," but has a different subject.

In Equal Groups, the roles of the factors differ. One factor is the number of objects in a group (like any quantity in addition and subtraction situations), and the other is a multiplier that indicates the number of groups. So, for example, 4 groups of 3 objects is arranged differently than 3 groups of 4 objects. Thus there are two kinds of division situations depending on which factor is the unknown (the number of objects in each group or the number of groups). In the Array situations, the roles of the factors do not differ. One factor tells the number of rows in the array, and the other factor tells the number of columns in the situation. But rows and columns depend on the orientation of the array. If an array is rotated 90°, the rows become columns and the columns become rows. This is useful for seeing the commutative property for multiplication[3.OA.5] in rectangular arrays and areas. This property can be seen to extend to Equal Group situations when Equal Group situations are related to arrays by arranging each group in a row and putting the groups under each other to form an array. Array situations can be seen as Equal Group situations if each row or column is considered as a group. Relating Equal Group situations to Arrays, and indicating rows or columns within arrays, can help students see that a corner object in an array (or a corner square in an area model) is not double counted: at a given time, it is counted as part of a row or as a part of a column but not both.

3.OA.5 Apply properties of operations as strategies to multiply and divide.

As noted in Table 3, row and column language can be difficult. The Array problems given in the table are of the simplest form in which a row is a group and Equal Groups language is used ("with 6 apples in each row"). Such problems are a good transition between the Equal Groups and array situations and can support the generalization of the commutative property discussed above. Problems in terms of "rows" and "columns," e.g., "The apples in the grocery window are in 3 rows and 6 columns," are difficult because of the distinction between the number of things *in a* row and the number *of* rows. There are 3 rows but the number of columns (6) tells how many are in each row. There are 6 columns but the number of rows (3) tells how many are in each column. Students do need to be able to use and understand these words, but this understanding can grow over time while students also learn and use the language in the other multiplication and division situations.

Variations of each type that use measurements instead of discrete objects are given in the Measurement and Data Progression. Grade 2 standards focus on length measurement[2.MD.1–4] and Grade 3 standards focus on area measurement.[3.MD.5–7] The measurement examples are more difficult than are the examples about discrete objects, so these should follow problems about discrete objects. Area problems where regions are partitioned by unit squares are foundational for Grade 3 standards because area is used as a model for single-digit multiplication and division strategies,[3.MD.7] in Grade 4 as a model for multi-digit multiplication and division and in Grade 5 and Grade 6 as a model for multiplication and division of decimals

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

2.MD.2 Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

2.MD.4 Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7 Relate area to the operations of multiplication and addition.

and of fractions.[5.NBT.6] The distributive property is central to all of these uses and will be discussed later.

The top row of Table 3 shows the usual order of writing multiplications of Equal Groups in the United States. The equation $3 \times 6 = \square$ means how many are in 3 groups of 6 things each: three sixes. But in many other countries the equation $3 \times 6 = \square$ means how many are 3 things taken 6 times (6 groups of 3 things each): six threes. Some students bring this interpretation of multiplication equations into the classroom. So it is useful to discuss the different interpretations and allow students to use whichever is used in their home. This is a kind of linguistic commutativity that precedes the reasoning discussed above arising from rotating an array. These two sources of commutativity can be related when the rotation discussion occurs.

5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Levels in problem representation and solution Multiplication and division problem representations and solution methods can be considered as falling within three levels related to the levels for addition and subtraction (see Appendix). Level 1 is making and counting all of the quantities involved in a multiplication or division. As before, the quantities can be represented by objects or with a diagram, but a diagram affords reflection and sharing when it is drawn on the board and explained by a student. The Grade 2 standards 2.OA.3 and 2.OA.4 are at this level but set the stage for Level 2. Standard 2.OA.3 relates doubles additions up to 20 to the concept of odd and even numbers and to counting by 2s (the easiest count-by in Level 2) by pairing and counting by 2s the things in each addend. 2.OA.4 focuses on using addition to find the total number of objects arranged in rectangular arrays (up to 5 by 5).

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Level 2 is repeated counting on by a given number, such as for 3: $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.

Supporting Level 2 methods with arrays

Small arrays (up to 5×5) support seeing and beginning to learn the Level 2 count-bys for the first five equal groups of the small numbers 2 through 5 if the running total is written to the right of each row (e.g., $3, 6, 9, 12, 15$). Students may write repeated additions and then count by ones without the objects, often emphasizing each last number said for each group. Grade 3 students can be encouraged to move as early as possible from equal grouping or array models that show all of the quantities to similar representations using diagrams that show relationships of numbers because diagrams are faster and less error-prone and support methods at Level 2 and Level 3. Some demonstrations of methods or of properties may need to fall back to initially showing all quantities along with a diagram.

The difficulty of saying and remembering the count-by for a given number depends on how closely related it is to 10, the base for our written and spoken numbers. For example, the count-by sequence for 5 is easy, but the count-by sequence for 7 is difficult. Decomposing with respect to a ten can be useful in going over a decade within a count-by. For example in the count-by for 7, students might use the following mental decompositions of 7 to compose up to and then go over the next decade, e.g., $14 + 7 = 14 + 6 + 1 = 20 + 1 = 21$. The count-by sequence can also be said with the factors, such as "one times three is *three*, two times three is *six*, three times three is *nine*, etc." Seeing as well as hearing the count-bys and the equations for

Composing up to, then over the next decade

7	14		21	28		35		42	49		56		63	70
		$6+1$			$2+5$		$5+2$			$1+6$		$4+3$		

There is an initial $3+4$ for $7+7$ that completes the reversing pattern of the partners of 7 involved in these mental decompositions with respect to the decades.

the multiplications or divisions can be helpful.

Level 3 methods use the associative property or the distributive property to compose and decompose. These compositions and decompositions may be additive (as for addition and subtraction) or multiplicative. For example, students multiplicatively compose or decompose:

> 4×6 is easier to count by 3 eight times:
>
> $$4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$$

Students may know a product 1 or 2 ahead of or behind a given product and say:

> I know 6×5 is 30, so 7×5 is $30 + 5$ more which is 35.
> This implicitly uses the distributive property:
>
> $$7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 3 = 30 + 5 = 35.$$

Students may decompose a product that they do not know in terms of two products they know (for example, 4×7 shown in the margin).

Students may not use the properties explicitly (for example, they might omit the second two steps), but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

The $5 + n$ pattern students used earlier for additions can now be extended to show how 6, 7, 8, and 9 times a number are $5 + 1$, $5 + 2$, $5 + 3$, and $5 + 4$ times that number. These patterns are particularly easy to do mentally for the numbers 4, 6, and 8. The 9s have particularly rich patterns based on $9 = 10 - 1$. The pattern of the tens digit in the product being 1 less than the multiplier, the ones digit in the product being 10 minus the multiplier, and that the digits in nines products sum to 9 all come from this pattern.

There are many opportunities to describe and reason about the many patterns involved in the Level 2 count-bys and in the Level 3 composing and decomposing methods. There are also patterns in multiplying by 0 and by 1. These need to be differentiated from the patterns for adding 0 and adding 1 because students often confuse these three patterns: $n + 0 = n$ but $n \times 0 = 0$, and $n \times 1$ is the pattern that does not change n (because $n \times 1 = n$). Patterns make multiplication by some numbers easier to learn than multiplication by others, so approaches may teach multiplications and divisions in various orders depending on what numbers are seen as or are supported to be easiest.

Multiplications and divisions can be learned at the same time and can reinforce each other. Level 2 methods can be particularly easy for division, as discussed above. Level 3 methods may be more difficult for division than for multiplication.

Throughout multiplication and division learning, students gain fluency and begin to know certain products and unknown factors.

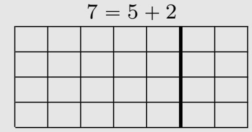
Decomposing 4×7

$$\begin{aligned} 4 \times 7 &= 4 \times (5 + 2) \\ &= (4 \times 5) + (4 \times 2) \\ &= 20 + 8 \\ &= 28 \end{aligned}$$

Supporting reasoning with area diagram

$7 = 5 + 2$

4



$20 + 8 = 28$

$4 \times 5 + 4 \times 2 = 4 \times 7$

The $5 + n$ pattern for multiplying the numbers 4, 6, and 8

n		$4 \times n$		$6 \times n$		$8 \times n$	
1	$5 + 1$	4	24	6	36	8	48
2	$5 + 2$	8	28	12	42	16	56
3	$5 + 3$	12	32	18	48	24	64
4	$5 + 4$	16	36	24	54	32	72
5	$5 + 5$	20	40	30	60	40	80

Patterns in multiples of 9

$1 \times 9 = 9$

$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$

$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$ etc

All of the understandings of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single-digit numbers and 10.[3.OA.7] Such fluency may be reached by becoming fluent for each number (e.g., the 2s, the 5s, etc.) and then extending the fluency to several, then all numbers mixed together. Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one-digit numbers. As should be clear from the foregoing, this isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning. All of the work on how different numbers fit with the base-ten numbers culminates in these "just know" products and is necessary for learning products. Fluent dividing for all single-digit numbers, which will combine just knows, knowing from a multiplication, patterns, and best strategy, is also part of this vital standard.

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Using a letter for the unknown quantity, the order of operations, and two-step word problems with all four operations Students in Grade 3 begin the step to formal algebraic language by using a letter for the unknown quantity in expressions or equations for one- and two-step problems.[3.OA.8] But the symbols of arithmetic, $\times$ or $\cdot$ or $*$ for multiplication and $\div$ or $/$ for division, continue to be used in Grades 3, 4, and 5.

3.OA.8 Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Understanding and using the associative and distributive properties (as discussed above) requires students to know two conventions for reading an expression that has more than one operation:

1. Do the operation inside the parentheses before an operation outside the parentheses (the parentheses can be thought of as hands curved around the symbols and grouping them).

2. If a multiplication or division is written next to an addition or subtraction, imagine parentheses around the multiplication or division (it is done before these operations). At Grades 3 through 5, parentheses can usually be used for such cases so that fluency with this rule can wait until Grade 6.

MP7 Making use of structure to make computation easier:

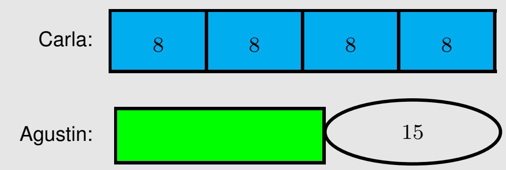
$$13 + 29 + 77 + 11 = (13 + 77) + (29 + 11)$$

These conventions are often called the Order of Operations and can seem to be a central aspect of algebra. But actually they are just simple "rules of the road" that allow expressions involving more than one operation to be interpreted unambiguously and thus are connected with the mathematical practice of communicating precisely.[MP6] Use of parentheses is important in displaying structure and thus is connected with the mathematical practice of making use of structure.[MP7] Parentheses are important in expressing the associative and especially the distributive properties. These properties are at the heart of Grades 3 to 5 because they are used in the Level 3 multiplication and division strategies, in multi-digit and decimal multiplication and division, and in all operations with fractions.

As with two-step problems at Grade 2,[2.OA.1, 2.MD.5] which involve only addition and subtraction, the Grade 3 two-step word problems vary greatly in difficulty and ease of representation. More difficult problems may require two steps of representation and solution rather than one. Use of two-step problems involving easy or middle difficulty adding and subtracting within 1,000 or one such adding or subtracting with one step of multiplication or division can help to maintain fluency with addition and subtraction while giving the needed time to the major Grade 3 multiplication and division standards.

A two-step problem with diagram showing problem situation and equations showing the two parts

Carla has 4 packages of silly bands. Each package has 8 silly bands in it. Agustin is supposed to get 15 fewer silly bands than Carla. How many silly bands should Agustin get?



C = number of Carla's silly bands
A = number of Agustin's silly bands

$$C = 4 \times 8 = 32$$

$$\begin{aligned} A + 15 &= C \\ A + 15 &= 32 \\ A &= 17 \end{aligned}$$

Students may be able to solve this problem without writing such equations.

Grade 4

Multiplication Compare Consider two diving boards, one 40 feet high, the other 8 feet high. Students in earlier grades learned to compare these heights in an additive sense—"This one is 32 feet higher than that one"—by solving additive Compare problems[2.OA.1] and using addition and subtraction to solve word problems involving length.[2.MD.5] Students in Grade 4 learn to compare these quantities multiplicatively as well: "This one is 5 times as high as that one."[4.OA.1,4.OA.2,4.MD.1,4.MD.2] In an additive comparison, the underlying question is *what amount would be added to one quantity* in order to result in the other. In a multiplicative comparison, the underlying question is *what factor would multiply one quantity* in order to result in the other. Multiplication Compare situations are shown in Table 3.

Language can be difficult in Multiplication Compare problems. The language used in the three examples in Table 3 is fairly simple, e.g., "A red hat costs 3 times as much as the blue hat." Saying the comparing sentence in the opposite way is more difficult. It could be said using division, e.g., "The cost of a red hat divided by 3 is the cost of a blue hat." It could also be said using a unit fraction, e.g., "A blue hat costs one-third as much as a red hat"; note however that multiplying by a fraction in not an expectation of the Standards in Grade 4. In any case, many languages do not use either of these options for saying the opposite comparison. They use the terms *three times more than* and *three times less than* to describe opposite multiplicative comparisons. These did not used to be acceptable usages in English because they mix the multiplicative and additive comparisons and are ambiguous. If the cost of a red hat is three times more than a blue hat that costs \$5, does a red hat cost \$15 (three times as much) or \$20 (three times more than: a difference that is three times as much)? However, the terms *three times more than* and *three times less than* are now appearing frequently in newspapers and other written materials. It is recommended to discuss these complexities with Grade 4 students while confining problems that appear on tests or in multi-step problems to the well-defined multiplication language in Table 3. The tape diagram for the additive Compare situation that shows a smaller and a larger tape can be extended to the multiplication Compare situation.

Fourth graders extend problem solving to multi-step word problems using the four operations posed with whole numbers. The same limitations discussed for two-step problems concerning representing such problems using equations apply here. Some problems might easily be represented with a single equation, and others will be more sensibly represented by more than one equation or a diagram and one or more equations. Numbers can be those in the Grade 4 standards, but the number of steps should be no more than three and involve only easy and medium difficulty addition and subtraction problems.

[2.OA.1] Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.OA.2] Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

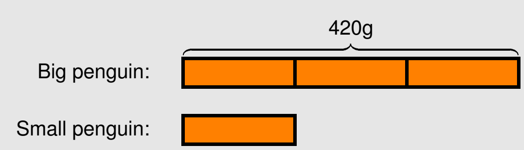
Tape diagram used to solve the Compare problem in Table 3

B is the cost of a blue hat in dollars
R is the cost of a red hat in dollars

[\$6]	$3 \times B = R$
[\$6][\$6][\$6]	$3 \times \$6 = \$18$

A tape diagram used to solve a Compare problem

A big penguin will eat 3 times as much fish as a small penguin. The big penguin will eat 420 grams of fish. All together, how much will the two penguins eat?



B = number of grams the big penguin eats
S = number of grams the small penguin eats

$$3 \cdot S = B$$
$$3 \cdot S = 420$$
$$S = 140$$

$$S + B = 140 + 420$$
$$= 560$$

Remainders In problem situations, students must interpret and use remainders with respect to context.[4.OA.3] For example, what is the smallest number of busses that can carry 250 students, if each bus holds 36 students? The whole number quotient in this case is 6 and the remainder is 34; the equation $250 = 6 \times 36 + 34$ expresses this result and corresponds to a picture in which 6 busses are completely filled while a seventh bus carries 34 students. Notice that the answer to the stated question (7) differs from the whole number quotient.

On the other hand, suppose 250 pencils were distributed among 36 students, with each student receiving the same number of pencils. What is the largest number of pencils each student could have received? In this case, the answer to the stated question (6) is the same as the whole number quotient. If the problem had said that the teacher got the remaining pencils and asked how many pencils the teacher got, then the remainder would have been the answer to the problem.

4.OA.3 Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Factors, multiples, and prime and composite numbers Students extend the idea of decomposition to multiplication and learn to use the term *multiple*.[4.OA.4] Any whole number is a multiple of each of its factors, so for example, 21 is a multiple of 3 and a multiple of 7 because $21 = 3 \cdot 7$. A number can be multiplicatively decomposed into equal groups and expressed as a product of these two factors (called factor pairs). A prime number has only one and itself as factors. A composite number has two or more factor pairs. Students examine various patterns in factor pairs by finding factor pairs for all numbers 1 to 100 (e.g., no even number other than 2 will be prime because it always will have a factor pair including 2). To find all factor pairs for a given number, students can search systematically, by checking if 2 is a factor, then 3, then 4, and so on, until they start to see a "reversal" in the pairs (for example, after finding the pair 6 and 9 for 54, students will next find the reverse pair, 9 and 6; all subsequent pairs will be reverses of previously found pairs). Students understand and use of the concepts and language in this area, but need not be fluent in finding all factor pairs. Determining whether a given whole number in the range 1 to 100 is a multiple of a given one-digit number is a matter of interpreting prior knowledge of division in terms of the language of multiples and factors.

4.OA.4 Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Generating and analyzing patterns This standard[4.OA.5] begins a small focus on reasoning about number or shape patterns, connecting a rule for a given pattern with its sequence of numbers or shapes. Patterns that consist of repeated sequences of shapes or growing sequences of designs can be appropriate for the grade. For example, students could examine a sequence of dot designs in which each design has 4 more dots than the previous one and they could reason about how the dots are organized in the design to determine the

4.OA.5 Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

total number of dots in the 100th design. In examining numerical sequences, fourth graders can explore rules of repeatedly adding the same whole number or repeatedly multiplying by the same whole number. Properties of repeating patterns of shapes can be explored with division. For example, to determine the 100th shape in a pattern that consists of repetitions of the sequence "square, circle, triangle," the fact that when we divide 100 by 3 the whole number quotient is 33 with remainder 1 tells us that after 33 full repeats, the 99th shape will be a triangle (the last shape in the repeating pattern), so the 100th shape is the first shape in the pattern, which is a square. Notice that the Standards do not require students to infer or guess the underlying rule for a pattern, but rather ask them to generate a pattern from a given rule and identify features of the given pattern.

Grade 5

As preparation for the Expressions and Equations Progression in the middle grades, students in Grade 5 begin working more formally with expressions.[5.OA.1, 5.OA.2] They write expressions to express a calculation, e.g., writing $2 \times (8+7)$ to express the calculation "add 8 and 7, then multiply by 2." They also evaluate and interpret expressions, e.g., using their conceptual understanding of multiplication to interpret $3 \times (18932+921)$ as being three times as large as $18932+921$, without having to calculate the indicated sum or product. Thus, students in Grade 5 begin to think about numerical expressions in ways that prefigure their later work with variable expressions (e.g., three times an unknown length is $3 \cdot L$). In Grade 5, this work should be viewed as exploratory rather than for attaining mastery; for example, expressions should not contain nested grouping symbols, and they should be no more complex than the expressions one finds in an application of the associative or distributive property, e.g., $(8+27)+2$ or $(6 \times 30)+(6 \times 7)$. Note however that the numbers in expressions need not always be whole numbers.

5.OA.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

Students extend their Grade 4 pattern work by working briefly with two numerical patterns that can be related and examining these relationships within sequences of ordered pairs and in the graphs in the first quadrant of the coordinate plane.[5.OA.3] This work prepares students for studying proportional relationships and functions in middle school.

5.OA.3 Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Connections to NF and NBT in Grades 3 through 5

Students extend their whole number work with adding and subtracting and multiplying and dividing situations to decimal numbers and fractions. Each of these extensions can begin with problems that include all of the subtypes of the situations in Tables 1 and 2. The operations of addition, subtraction, multiplication, and division continue to be used in the same way in these problem situations when they are extended to fractions and decimals (although making these extensions is not automatic or easy for all students). The connections described for Kindergarten through Grade 3 among word problem situations, representations for these problems, and use of properties in solution methods are equally relevant for these new kinds of numbers. Students use the new kinds of numbers, fractions and decimals, in geometric measurement and data problems and extend to some two-step and multi-step problems involving all four operations. In order to keep the difficulty level from becoming extreme, there should be a tradeoff between the algebraic or situational complexity of any given problem and its computational difficulty taking into account the kinds of numbers involved.

As students' notions of quantity evolve and generalize from discrete to continuous during Grades 3–5, their notions of multiplication evolves and generalizes. This evolution deserves special attention because it begins in OA but ends in NF. Thus, the concept of multiplication begins in Grade 3 with an entirely discrete notion of "equal groups."[3.OA.1] By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion "times as much."[4.OA.1] This notion has more affinity to continuous quantities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as $\frac{3}{4}$ cup of flour.[4.NF.4,4.MD.2] By Grade 5, when students multiply fractions in general,[5.NF.4] products can be larger or smaller than either factor, and multiplication can be seen as an operation that "stretches or shrinks" by a scale factor.[5.NF.5] This view of multiplication as scaling is the appropriate notion for reasoning multiplicatively with continuous quantities.

3.OA.1 Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

4.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

5.NF.5 Interpret multiplication as scaling (resizing), by:

Where the Operations and Algebraic Thinking Progression is heading

Connection to the Number System The properties of and relationships between operations that students worked with in Grades K–5 will become even more prominent in extending arithmetic to systems that include negative numbers; meanwhile the meanings of the operations will continue to evolve, e.g., subtraction will become "adding the opposite."

Connection to Expressions and Equations In Grade 6, students will begin to view expressions not just as calculation recipes but as entities in their own right, which can be described in terms of their parts. For example, students see $8 * (5+2)$ as the product of 8 with the sum $5+2$. In particular, students must use the conventions for order of operations to *interpret* expressions, not just to evaluate them. Viewing expressions as entities created from component parts is essential for seeing the structure of expressions in later grades and using structure to reason about expressions and functions.

As noted above, the foundation for these later competencies is laid in Grade 5 when students write expressions to record a "calculation recipe" without actually evaluating the expression, use parentheses to formulate expressions, and examine patterns and relationships numerically and visually on a coordinate plane graph.[5.OA.1,5.OA.2] Before Grade 5, student thinking that also builds toward the Grade 6 EE work is focusing on the expressions on each side of an equation, relating each expression to the situation, and discussing the situational and mathematical vocabulary involved to deepen the understandings of expressions and equations.

[5.OA.1] Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

[5.OA.2] Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

In Grades 6 and 7, students begin to explore the systematic algebraic methods used for solving algebraic equations. Central to these methods are the relationships between addition and subtraction and between multiplication and division, emphasized in several parts of this Progression and prominent also in the 6–8 Progression for the Number System. Students' varied work throughout elementary school with equations with unknowns in all locations and in writing equations to decompose a given number into many pairs of addends or many pairs of factors are also important foundations for understanding equations and for solving equations with algebraic methods. Of course, any method of solving, whether systematic or not, relies on an understanding of what solving itself is—namely, a process of answering a question: which values from a specified set, if any, make the equation true?[6.EE.5]

[6.EE.5] Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Students represent and solve word problems with equations involving one unknown quantity in K through 5. The quantity was expressed by a □ or other symbol in K–2 and by a letter in Grades 3 to 5. Grade 6 students continue the K–5 focus on representing a problem situation using an equation (a situation equation) and then (for the more difficult situations) writing an equivalent equation that is easier to solve (a solution equation). Grade 6 students discuss their reasoning more explicitly by focusing on the structures of expressions and using the properties of operations explicitly. Some of the math drawings that students have used in K through 5 to represent problem situations continue to be used in the middle grades. These can help students throughout the grades deepen the connections they make among the situation and problem representations by a drawing and/or by an equation, and support the informal K–5 and increasingly formal 6–8 solution methods arising from understanding the structure of expressions and equations.

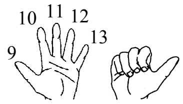
Appendix. Methods used for solving single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away.

Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Adding ($8 + 6 = \square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown.

Subtracting ($14 - 8 = \square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown added. Use this strategy for Take From/Result Unknown.

Levels	**8 + 6 = 14**	**14 – 8 = 6**
Level 1: Count all	Count All a 1 2 3 4 5 6 7 8 b 1 2 3 4 5 6 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b c
Level 2: Count on	Count On 8 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  9 10 11 12 13 14 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten 10 + 4 10 + 4	14 – 8: I make a ten for 8 + ? = 14 8 + 2 + 4 6 8 + 6 = 14
Doubles ± *n*	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2. Counting On.

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on can be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words and extended experience with Level 1 methods in Kindergarten.

> Adding (e. g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
>
> Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The keeping track method is monitored so that counting on stops when the known total has been reached. The keeping track method tells the unknown addend.
>
> Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$ and uses counting on to find an unknown addend (as above).

The problems in Table 2 which are solved by Level 1 methods in Kindergarten can also be solved using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting).

The middle difficulty (lightly shaded) problem types in Table 2 for Grade 1 are directly accessible with the embedded thinking of Level 2 methods and can be solved by counting on.

> Finding an unknown addend (e.g., $8 + \square = 14$) is used for Add To/Change Unknown, Put Together/Take Apart/Addend Unknown, and Compare/Difference Unknown. It is also used for Take From/Change Unknown ($14 - \square =$

8) after a student has decomposed the total into two addends, which means they can represent the situation as $14 - 8 = \square$.

Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in Table 2). Grade 1 students do not necessarily master the Compare Bigger Unknown or Smaller Unknown problems with the misleading language in the bottom row of Table 2.

Solving an equation such as $6 + 8 = \square$ by counting on from 8 relies on the understanding that $8 + 6$ gives the same total, an implicit use of the commutative property without the accompanying written representation $6 + 8 = 8 + 6$.

Level 3. Convert to an Easier Equivalent Problem.

Decompose an addend and compose a part with another addend.

These methods can be used to add or to find an unknown addend (and thus to subtract). These methods implicitly use the associative property.

Adding

Make a ten. E.g, for $8 + 6 = \square$,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so $8 + 6$ becomes $10 + 4$.

Doubles plus or minus 1. E.g., for $6 + 7 = \square$,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so $6 + 7$ becomes $12 + 1$.

Finding an unknown addend

Make a ten. E. g., for $8 + \square = 14$,

$$8 + \underline{2} = 10 \text{ and } \underline{4} \text{ more makes } 14.\ \underline{2 + 4} = 6.$$

So $8 + \square = 14$ is done as two steps: how many up to ten and how many over ten (which can be seen in the ones place of 14).

Doubles plus or minus 1. E.g., for $6 + \square = 13$,

$$6 + \underline{6 + 1} = 12 + 1.\ \underline{6 + 1} = 7.$$

So $6 + \square = 13$ is done as two steps: how many up to 12 $(6 + 6)$ and how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend. E.g., solve $14-8=\square$ or $13-6=\square$ as $8+\square=14$ or $6+\square=13$ by the above methods (make a ten or doubles plus or minus 1).

Make a ten by going down over ten. E.g., $14-8=\square$ can be done in two steps by going down over ten: $14-4$ (to get to 10) $-4=6$.

The Level 1 and Level 2 problem types can be solved using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend as shown above by methods at any level, but usually at Level 2 or 3. Many students only show in their writing part of this multi-step process of re-representing the situation.

> Students re-represent Add To/Start Unknown $\square+6=14$ situations as $6+\square=14$ by using the commutative property (formally or informally).

> Students re-represent Take From/Start Unknown $\square-8=6$ situations by reversing as $6+8=\square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare misleading language situations can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct solution by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same diagrams used for the other versions of compare situations; focusing on which quantity is bigger and which is smaller helps to overcome the misleading language.

Some students may solve Level 3 problem types by doing the above re-representing but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, facilitating such re-representing. Labels on diagrams can help connect the parts of the diagram to the corresponding parts of the situation. But students may know and understand things that they may not use for a given solution of a problem as they increasingly do various representing and re-representing steps mentally.

Progressions for the Common Core State Standards in Mathematics (draft)

6 March 2015

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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

Number and Operations in Base Ten, K–5

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding in computations, decomposing 1 ten into 10 ones and composing a ten from 10 ones.

The power of the base-ten system is in repeated bundling by ten: 10 tens make a unit called a hundred. Repeating this process of creating new units by bundling in groups of ten creates units called *thousand, ten thousand, hundred thousand* . . . In learning about decimals, children partition a one into 10 equal-sized smaller units, each of which is a tenth. Each base-ten unit can be understood in terms of any other base-ten unit. For example, one hundred can be viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths. Algorithms• for operations in base ten draw on such relationships among the base-ten units.

Computations Standard algorithms• for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit.

Beginning in Kindergarten, the requisite abilities develop gradually over the grades. Experience with addition and subtraction within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2 standard.[2.OA.2] Computations within 20 that "cross 10," such as $9 + 8$ or $13 - 6$, are especially relevant to NBT because they afford the development of the Level 3 make-a-ten strategies for addition and subtraction described in the OA Progression. From the NBT perspective, make-a-ten strategies are (implicitly) the first instances of composing or decomposing a base-ten unit. Such strategies are a foundation for understanding in Grade 1 that addition may require composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve decomposing a ten.[2.NBT.7]

Strategies and algorithms The Standards distinguish strategies• from algorithms. Work with computation begins with use of strategies and "efficient, accurate, and generalizable methods." (See Grade 1 critical areas 1 and 2, Grade 2 critical area 2; Grade 4 critical area 1.) For each operation, the culmination of this work is signaled in the Standards by use of the term "standard algorithm."

Initially, students compute using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction (or multiplication and division). They relate their strategies to written methods and explain the reasoning used (for addition within 100 in Grade 1; for addition and subtraction within 1000 in Grade 2) or illustrate and explain their calculations with equations, rectangular arrays, and/or area models (for multiplication and division in Grade 4).

Students' initial experiences with computation also include development, discussion, and use of "efficient, accurate, and generalizable methods." So from the beginning, students see, discuss, and explain methods that can be generalized to all numbers represented in the base-ten system. Initially, they may use written methods that include extra helping steps to record the underlying reasoning. These helping step variations can be important initially for under-

• From the Standards glossary:

> **Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.

In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. This progression gives examples of different recording methods and discusses their advantages and disadvantages.

• The Standards do not specify a particular standard algorithm for each operation. This progression gives examples of algorithms that could serve as the standard algorithm and discusses their advantages and disadvantages.

[1.OA.6] Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

[2.OA.2] Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

[1.NBT.4] Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

[2.NBT.7] Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

• From the Standards glossary:

> **Computation strategy.** Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Examples of computation strategies are given in this progression and in the Operations and Algebraic Thinking Progression.

standing. Over time, these methods can and should be abbreviated into shorter written methods compatible with fluent use of standard algorithms.

Students may also develop and discuss mental or written calculation methods that cannot be generalized to all numbers or are less efficient than other methods.

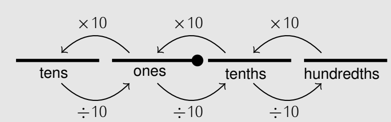
Mathematical practices The Standards for Mathematical Practice are central in supporting students' progression from understanding and use of strategies to fluency with standard algorithms. The initial focus in the Standards on understanding and explaining such calculations, with the support of visual models, affords opportunities for students to see mathematical structure as accessible, important, interesting, and useful.

Students learn to see a number as composed of its base-ten units (MP.7). They learn to use this structure and the properties of operations to reduce computing a multi-digit sum, difference, product, or quotient to a collection of single-digit computations in different base-ten units. (In some cases, the Standards refer to "multi-digit" operations rather than specifying numbers of digits. The intent is that sufficiently many digits should be used to reveal the standard algorithm for each operation in all its generality.) Repeated reasoning (MP.8) that draws on the uniformity of the base-ten system is a part of this process. For example, in addition computations students generalize the strategy of making a ten to composing 1 base-ten unit of next-highest value from 10 like base-ten units.

Students abstract quantities in a situation (MP.2) and use concrete models, drawings, and diagrams (MP.4) to help conceptualize (MP.1), solve (MP.1, MP.3), and explain (MP.3) computational problems. They explain correspondences between different methods (MP.1) and construct and critique arguments about why those methods work (MP.3). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten?, and to probe into the referents for symbols used (MP.2), e.g., does that 1 represent the number of apples in the problem?

Some methods may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, comparing methods offers opportunities to raise the topic of using appropriate tools strategically (MP.5). Comparing methods can help to illustrate the advantages of standard algorithms: standard algorithms are general methods that minimize the number of steps needed and, once, fluency is achieved, do not require new reasoning.

Uniformity of the base-ten system



For any base-ten unit, 10 copies compose 1 base-ten unit of next-highest value, e.g., 10 ones are 1 ten, 10 tens are 1 hundred, etc.

Kindergarten

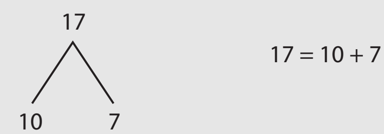
In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1 + 9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place value[K.NBT.1] Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

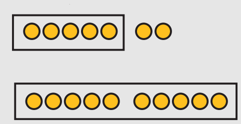
• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

Number-bond diagram and equation



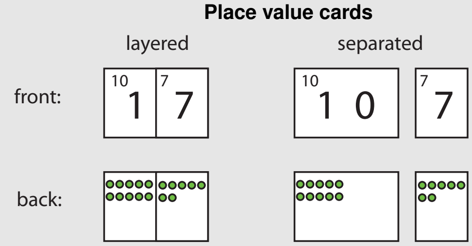
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames



Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

Place value cards



Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings. When any of the number arrangements is turned over, the one card is hidden under the tens card. Children can see this and that they need to move the ones dots above and on the right side of the tens card.

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, ..., nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section.

The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones).

By working with teen numbers in this way in Kindergarten, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

Grade 1

In first grade, students learn to view ten ones as a unit called a ten. The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and they add and subtract using this understanding.

Extend the counting sequence and understand place value Through structured learning time, discussion, and practice students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers 10, . . . , 90, in written and in spoken form, as 1 ten, . . . , 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones. Saying 67 as "6 tens, 7 ones" as well as "sixty-seven" can help students focus on the tens and ones structure of written numerals.

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols >, =, and <. Correctly placing the < and > symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.

Use place value understanding and properties of operations to add and subtract First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or

Part of a numeral list

91	101	111
92	102	112
93	103	113
94	104	114
95	105	115
96	106	116
97	107	117
98	108	118
99	109	119
100	110	120

In the classroom, a list of the numerals from 1 to 120 can be shown in columns of 10 to help highlight the base-ten structure, e.g., in the leftmost column, the 9s (indicating 9 tens) are lined up and the ones increase by 1 from 91 to 99. The numbers 101, . . . , 120 may be especially difficult for children to write.

[1.NBT.2]Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

a 10 can be thought of as a bundle of ten ones—called a "ten."

b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

[1.NBT.3]Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

[1.NBT.4]Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones. In particular, showing composition of a ten with objects or drawings affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

Combining tens and ones separately as illustrated in the margin can be extended to the general method of combining like base-ten units. The margin illustrates combining ones, then tens. Like base-ten units can be combined in any order, but going from smaller to larger eliminates the need to go back to a given place to add in a new unit. For example, in computing $46 + 37$ by combining tens, then ones (going left to right), one needs to go back to add in the new 1 ten: "4 tens and 3 tens is 7 tens, 6 ones and 7 ones is 13 ones which is 1 ten and 3 ones, 7 tens and 1 ten is 8 tens. The total is 8 tens and 3 ones: 83."

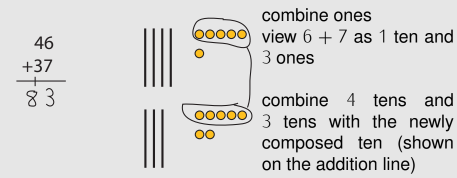
Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings of 5-groups can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as $70 - 40$ can be viewed as 7 tens minus 4 tens and represented with concrete models such as objects bundled in tens or drawings. Children use the relationship between subtraction and addition when they view $80 - 70$ as an unknown addend addition problem, $70 + \square = 80$, and reason that 1 ten must be added to 70 to make 80, so $80 - 70 = 10$.

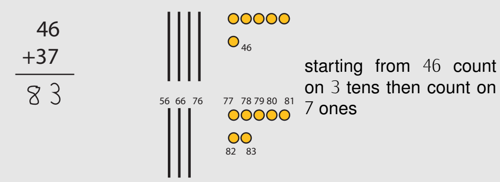
First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases. This helps students to avoid making the generalization "in each column, subtract the larger digit from the smaller digit, independent of whether the larger digit is in the subtrahend or minuend," e.g., making the error 82 - 45 = 43.

Adding tens and ones separately



This method is an application of the commutative and associative properties. The diagrams can help children with understanding and explaining the steps (MP.1). Advantages of writing the 1 below the addends are discussed in the Grade 2 margin.

Counting on by tens



Counting on by tens from 46, beginning 56, 66, 76, then counting on by ones. This method can be generalized, but the complexity of the counting on required and the lack of efficiency becomes apparent as the number of digits in the addends increases.

[1.NBT.5] Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

[1.NBT.6] Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

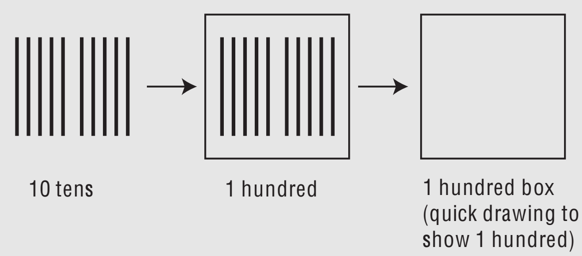
At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

[2.NBT.1a] Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a 100 can be thought of as a bundle of ten tens—called a "hundred."

Drawings to support seeing 10 tens as 1 hundred



Representations such as manipulative materials, math drawings and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3] Unlayering three-digit place value cards like the two-digit cards shown for Kindergarten and Grade 1 reveals the expanded form of the number.

[2.NBT.3] Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number said after 499 or reached after 500 counts of 1.

A major task for Grade 2 is learning the counting sequence from 100 to 1,000. As part of learning and using the base-ten structure, students count by ones within various parts of this sequence, especially the more difficult parts that "cross" tens or hundreds.

Building on their place value work, students continue to develop proficiency with mental computation.[2.NBT.8] They extend this to skip-counting by 5s, 10s, and 100s to emphasize and experience the tens and hundreds within the sequence and to prepare for multiplication.[2.NBT.2]

[2.NBT.8] Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

[2.NBT.2] Count within 1000; skip-count by 5s, 10s, and 100s.

Comparing magnitudes of two-digit numbers uses the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers uses the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4] Drawings help support these understandings.

[2.NBT.4] Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, =, and $<$ symbols to record the results of comparisons.

[2.NBT.5] Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

[2.NBT.6] Add up to four two-digit numbers using strategies based on place value and properties of operations.

Use place value understanding and properties of operations to add and subtract Students fluently add and subtract within 100.[2.NBT.5] They also add and subtract within 1000.[2.NBT.7] They explain why addition and subtraction strategies work, using place value and the properties of operations, and may support their explanations with drawings or objects.[2.NBT.9] Because adding and subtracting within 100 is a special case of adding and subtracting within 1000, methods within 1000 will be discussed before fluency within 100.

Two written methods for addition within 1000 are shown in the margins of this page and the next. The first explicitly shows the hundreds, tens, and ones that are being added; this can be helpful conceptually to students. The second method, shown on the next page, explicitly shows the adding of the single digits in each place and how this approach can continue on to places on the left.

Drawings can support students in explaining these and other methods. The drawing in the margin shows addends decomposed into their base-ten units (here, hundreds, tens, and ones), with the tens and hundreds represented by quick drawings. These quick drawings show each hundred as a single unit rather than ten tens (see illustration on p. 8), generalizing the approach that students used in Grade 1 of showing a ten as a single unit rather than as 10 separate ones. The putting together of like quick drawings illustrates adding like units as specified in 2.NBT.7: add ones to ones, tens to tens, and hundreds to hundreds. The drawing also shows newly composed units. Steps of adding like units and composing new units shown in the drawing can be connected with corresponding steps in other written methods. This also facilitates discussing how different written methods may show steps in different locations or different orders (MP.1 and MP.3). The associative and the commutative properties enable adding like units to occur.

The first written method is a helping step variation that generalizes to all numbers in base ten but becomes impractical because of writing so many zeros. Students can move from this method to the second method (or another compact method) by seeing how the steps of the two methods are related. Some students might make this transition in Grade 2, some in Grade 3, but all need to make it by Grade 4 where fluency requires a more compact method.

This first method can be seen as related to oral counting-on or written adding-on methods in which an addend is decomposed into hundreds, tens, and ones. These are successively added to the other addend, with the student saying or writing successive totals. These methods require keeping track of what parts of the decomposed addend have been added, and skills of mentally counting or adding hundreds, tens, and ones correctly. For example, beginning with hundreds: 278 plus 100 is 378 ("I've used all of the hundreds"), 378 plus 30 is 408 and plus 10 (to add on all of the 40) is 418, and 418 plus 7 is 425. One way to keep track: draw the 147 and cross out parts as they are added on. Counting-on and adding-on methods become even more difficult with numbers over 1000. If they arise

[2.NBT.5] Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

[2.NBT.7] Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

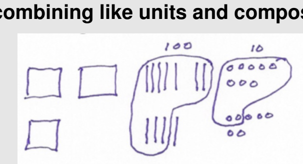
[2.NBT.9] Explain why addition and subtraction strategies work, using place value and the properties of operations.[2]

Addition: Recording newly composed units in separate rows

278	278	278	278
+ 147	+ 147	+ 147	+ 147
	300	300	300
		110	110
			15
			425

The computation shown proceeds from left to right, but could have gone from right to left. Working from left to right has two advantages: Many students prefer it because they read from left to right; working first with the largest units yields a closer approximation earlier.

Illustrating combining like units and composing new units



The drawing shows the base-ten units of 278 and 147. Like units are shown together, with boundaries drawn around ten tens and ten ones to indicate the newly composed hundred and the newly composed ten. The newly composed units could also be indicated by crossing out grouped units and drawing a single next-highest unit, e.g., crossing out the group of ten ones and drawing a single ten. Drawings like this can be used to illustrate and explain both of the written computations below.

from students, they should be discussed. But the major focus for addition within 1000 needs to be on methods such as those in the margin that are simpler for students and lead toward fluency (e.g., recording new units in separate rows shown) or are sufficient for fluency (e.g., recording new units in one row).

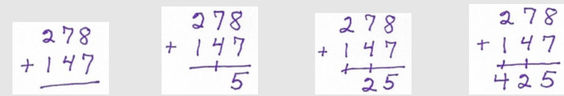
Drawings and steps for a generalizable method of subtracting within 1000 are shown in the margin. The total 425 does not have enough tens or ones to subtract the 7 tens or 8 ones in 278. Therefore one hundred is decomposed to make ten tens and one ten is decomposed to make ten ones. These decompositions can be done and written in either order; starting from the left is shown because many students prefer to operate in that order. In the middle step, one hundred has been decomposed (making 3 hundreds, 11 tens, 15 ones) so that the 2 hundreds 7 tens and 8 ones in 278 can be subtracted. These subtractions of like units can also be done in any order. When students alternate decomposing and subtracting like units, they may forget to decompose entirely or in a given column after they have just subtracted (e.g., after subtracting 8 from 15 to get 7, they move left to the tens column and see a 1 on the top and a 7 on the bottom and write 6 because they are in subtraction mode, having just subtracted the ones).

Students can also subtract within 1000 by viewing a subtraction as an unknown addend problem, e.g., $278 + ? = 425$. Counting-on and adding-on methods such as those described above for addition can be used. But as with addition, the major focus needs to be on methods that lead toward fluency or are sufficient for fluency (e.g., recording as shown in the second row in the margin).

In Grade 1, students have added within 100 using concrete models or drawings and used at least one method that is generalizable to larger numbers (such as between 101 and 1000). In Grade 2, they can make that generalization, using drawings for explanation as discussed above. This extension could be done first for two-digit numbers (e.g., $78 + 47$) so that students can see and discuss composing both ones and tens without the complexity of hundreds in the drawings or numbers (imagine the margin examples for $78+47$). After computing totals that compose both ones and tens for two-digit numbers, then within 1000, the type of problems required for fluency in Grade 2 seem easy, e.g., $28 + 47$ requires only composing a new ten from ones. This is now easier to do without drawings: one just records the new ten before it is added to the other tens or adds it to them mentally.

A similar approach can be taken for subtraction: first using concrete models or drawings to solve subtractions within 100 that involve decomposing one ten, then rather quickly solving subtractions that require two decompositions. Spending a long time on subtraction within 100 can stimulate students to count on or count down, which, as discussed above, are methods that are considerably more difficult with numbers above 100. Problems with different types of decompositions could be included so that students solve problems

Addition: Recording newly composed units in the same row



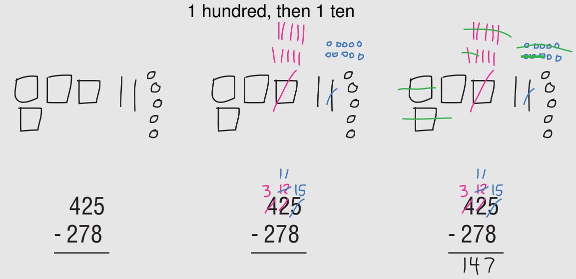
	Add the ones, $8 + 7$, and record these 15 ones with 1 on the line in the tens column and 5 below in the ones place.	Add the tens, $7 + 4 + 1$, and record these 12 tens with 1 on the line in the hundreds column and 2 below in the tens place.	Add the hundreds, $2 + 1 + 1$, and record these 4 hundreds below in the hundreds column.

Digits representing newly composed units are placed below the addends, on the line. This placement has several advantages. Each two-digit partial sum (e.g., "15") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 5) rather than "write the 5 and carry the 1" (write 5, then 1).

Subtraction: Decomposing where needed first

decomposing left to right, 1 hundred, then 1 ten

now subtract



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

requiring two, one, and no decompositions. Then students can spend time on subtractions that include multiple hundreds (totals from 201 to 1000). Relative to these experiences, the objectives for fluency at this grade are easy: focusing within 100 just on the two cases of one decomposition (e.g., $73 - 28$) or no decomposition (e.g., $78 - 23$) without drawings.

Students also add up to four two-digit numbers using strategies based on place value and properties of operations.[2.NBT.6] This work affords opportunities for students to see that they may have to compose more than one ten, and as many as three new tens. It is also an opportunity for students to reinforce what they have learned by informally using the commutative and associative properties. They could mentally add all of the ones, then write the new tens in the tens column, and finish the computation in writing. They could successively add each addend or add the first two and last two addends and then add these totals. Carefully chosen problems could suggest strategies that depend on specific numbers. For example, $38+47+93+62$ can be easily added by adding the first and last numbers to make 100, adding the middle two numbers to make 140, and increasing 140 by 100 to make 240. Students also can develop special strategies for particularly easy computations such as $398 + 529$, where the 529 gives 2 to the 398 to make 400, leaving 400 plus 527 is 927. But the major focus in Grade 2 needs to remain on the methods that work for all numbers and generalize readily to numbers beyond 1000.

[2.NBT.6] Add up to four two-digit numbers using strategies based on place value and properties of operations.

Grade 3

At Grade 3, the major focus is multiplication,• so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

• See the progression on Operations and Algebraic Thinking.

Use place value understanding and properties of operations to perform multi-digit arithmetic Students fluently add and subtract within 1000 using methods based on place value, properties of operations, and/or the relationship between addition and subtraction.[3.NBT.2] They focus on methods that generalize readily to larger numbers so that these methods can be extended to 1,000,000 in Grade 4 and fluency can be reached with such larger numbers. Fluency within 1000 implies that students use written methods without concrete models or drawings, though concrete models or drawings can be used with explanations to overcome errors and to continue to build understanding as needed.

[3.NBT.2] Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students and often is sufficient for practical purposes. Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

[3.NBT.1] Use place value understanding to round whole numbers to the nearest 10 or 100.

The special role of 10 in the base-ten system is important in understanding multiplication of one-digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication: $3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150$. It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non-zero digits, then shift the product one place to the left to make the result ten times as large.

[3.NBT.3] Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

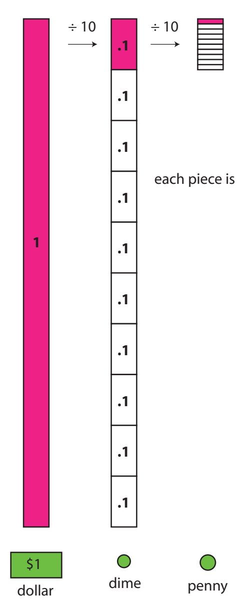
Grade 4

At Grade 4, students extend their work in the base-ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi-digit whole numbers In the base-ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (thousand, million, billion, trillion, etc.). Thus, 457,000 is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in previous grades apply to these numbers, because of the uniformity of the base-ten system.[4.NBT.3]

Decimal notation and fractions Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base-ten system to non-whole numbers.

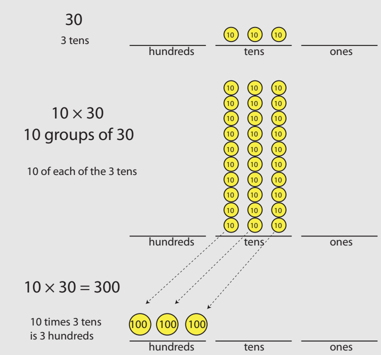


Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$. As with whole-number expansions in the base-ten system, each unit in this decomposition is ten times the unit to its right, reflecting the uniformity of the base-ten system. This can be connected with the use of base-ten notation to represent $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

[4.NBT.1] Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 represented as 3 tens each taken 10 times



Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

[4.NBT.2] Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using $>$, =, and $<$ symbols to record the results of comparisons.

[4.NBT.3] Use place value understanding to round multi-digit whole numbers to any place.

[4.NF.5] Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[3]

[4.NF.6] Use decimal notation for fractions with denominators 10 or 100.

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, =, or $<$, and justify the conclusions, e.g., by using a visual model.

However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place, as illustrated in the margin.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins 3.1415 . . .)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

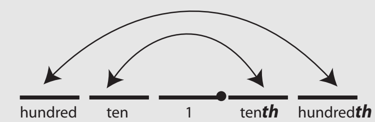
Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students understand how 2 tenths and 7 hundredths make 27 hundredths. Place value cards can be layered with the places farthest from the decimal point on the bottom (see illustration of the whole number cards on p. 5). These places are then covered by each place toward the decimal point: Tenths go on top of hundredth, and tens go on top of hundreds (for example, .2 goes on top of .07 to make .27, and 20 goes on top of 700 to make 720).

Use place value understanding and properties of operations to perform multi-digit arithmetic Students fluently add and subtract multi-digit numbers through 1,000,000 using the standard algorithm.[4.NBT.4] Because students in Grade 2 and Grade 3 have been using at least one method that readily generalizes to 1,000,000, this extension does not have to take a long time. Thus, students will have time for the major NBT focus for this grade: multiplication and division.

In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose, which is why 4.NBT.5 explicitly states that they are to be used to illustrate and explain the calculation. By reasoning repeatedly (MP.8) about the connection between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

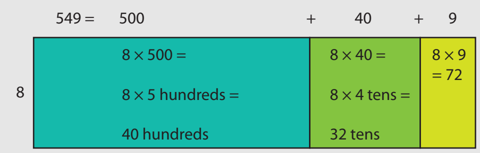
Symmetry with respect to the ones place



[4.NBT.4] Fluently add and subtract multi-digit whole numbers using the standard algorithm.

[4.NBT.5] Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Multiplication: Illustrating partial products with an area model

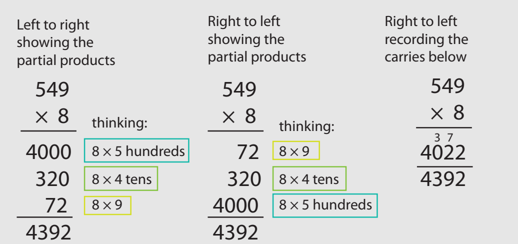


Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

An area model can be used for any multiplication situation after students have discussed how to show an equal groups or a compare situation with an area model by making the length of the rectangle represent the size of the equal groups or the larger compared quantity imagining things inside the square units to make an array (but not drawing them), and understanding that the dimensions of the rectangle are the same as the dimensions of the imagined array, e.g., an array illustrating 8 x 549 would have 8 rows and 549 columns. (See the Operations and Algebraic Thinking Progression for discussion of "equal groups" and "compare" situations.)

Multiplication: Recording methods



The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549*, thus keeping the digits of the products close to each other, e.g., the* 7 *from* $8 \times 9 = 72$ *is written diagonally to the left of the* 2 *rather than above the* 4 *in* 549*. The colors indicate correspondences with the area model above.*

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

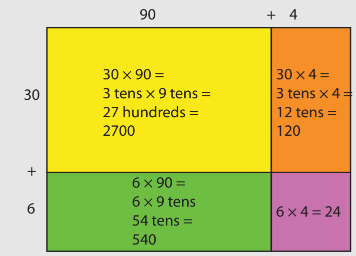
Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$\begin{aligned} 36 \times 94 &= (30+6) \times (94) \\ &= 30 \times 94 + 6 \times 94 \\ &= 30 \times (90+4) + 6 \times (90+4) \\ &= 30 \times 90 + 30 \times 4 + 6 \times 90 + 6 \times 4. \end{aligned}$$

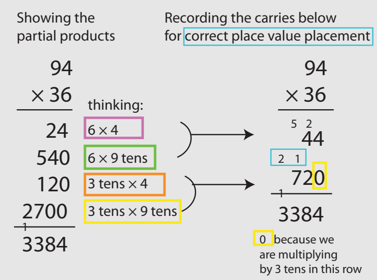
The four products in the last line correspond to the four rectangles in the area model in the margin. Their factors correspond to the factors in written methods. When written methods are abbreviated, some students have trouble seeing how the single-digit factors are related to the two-digit numbers whose product is being computed (MP.2). They may find it helpful initially to write each two-digit number as the sum of its base-ten units (e.g., writing next to the calculation $94 = 90 + 4$ and $36 = 30 + 6$) so that they see what the single digits are. Some students also initially find it helpful to write what they are multiplying in front of the partial products (e.g., $6 \times 4 = 24$). These helping steps can be dropped when they are no longer needed. At any point before or after their acquisition of fluency, some students may prefer to multiply from the left because they find it easier to align the subsequent products under this biggest product.

Illustrating partial products with an area model



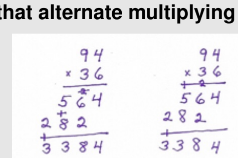
The products of base-ten units are shown as parts of a rectangular region. Such area models can support understanding and explaining of different ways to record multiplication. For students who struggle with the spatial demands of other methods, a useful helping step method is to make a quick sketch like this with the lengths labeled and just the partial products, then to add the partial products outside the rectangle.

Methods that compute partial products first



These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. *The digits* 2 *and* 1 *are surrounded by a blue box. The* 1 *from* $30 \times 4 = 120$ *is placed correctly in the hundreds place and the digit* 2 *from* $30 \times 90 = 2700$ *is placed correctly in the thousands place. If these digits had been placed above* 94, *they would be in incorrect places. Note that the* 0 *(surrounded by a yellow box) in the ones place of the second row of the method on the right is there because the whole row of digits is produced by multiplying by* 30 *(not* 3*). Colors on the left correspond with the area model above.*

Methods that alternate multiplying and adding



These methods put the newly composed units from a partial product in the correct column, then they are added to the next partial product. These alternating methods are more difficult than the methods above that show the four partial products. The first method can be used in Grade 5 division when multiplying a partial quotient times a two-digit divisor.

Not shown is the recording method in which the newly composed units are written above the top factor (e.g., 94). This puts the hundreds digit of the tens times ones product in the tens column (e.g., the 1 hundred in 120 from 30×4 *above the 9 tens in 94). This placement violates the convention that students have learned: a digit in the tens place represents tens, not hundreds.*

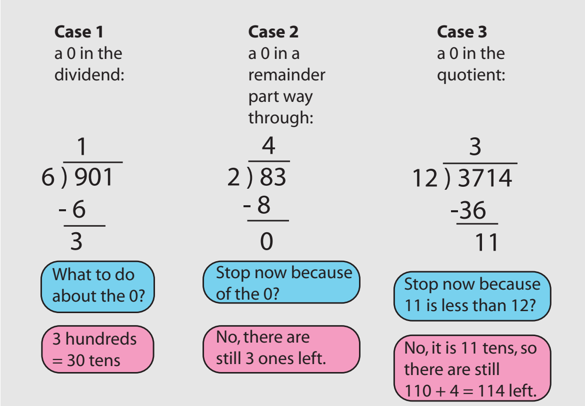
General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. See the figure in the margin. As with multiplication, this relies on the distributive property. This can be viewed as finding the side length of a rectangle (the divisor is the length of the other side) or as allocating objects (the divisor is the number of groups or the number of objects in each group). See the figure on the next page for an example.

Multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$.• In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation.

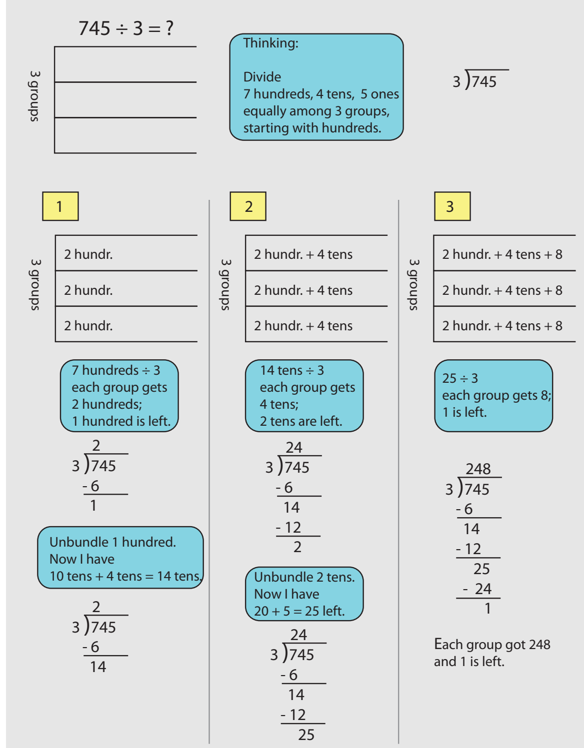
Cases involving 0 in division may require special attention. See the figure below.

Cases involving 0 in division



[4.NBT.6] Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Division as finding group size



$745 \div 3$ can be viewed as allocating 745 objects bundled in 7 hundreds, 4 tens, and 3 ones equally among 3 groups. In Step 1, the 2 indicates that each group got 2 hundreds, the 6 is the number of hundreds allocated, and the 1 is the number of hundreds not allocated. After Step 1, the remaining hundred is decomposed as 10 tens and combined with the 4 tens (in 745) to make 14 tens.

• A note on notation

The result of division within the system of whole numbers is frequently written as:

$$84 \div 10 = 8\,R\,4 \quad \text{and} \quad 44 \div 5 = 8\,R\,4.$$

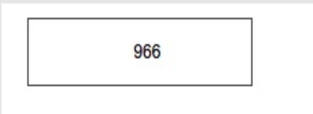
Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation $8\,R\,4$ does not indicate a number.

Rather than writing the result of division in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written as:

$$84 = 8 \times 10 + 4 \quad \text{and} \quad 44 = 8 \times 5 + 4.$$

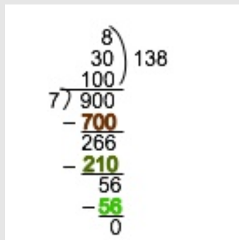
Division as finding side length

? hundreds + ? tens + ? ones

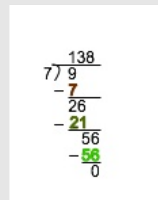


Find the unknown length of the rectangle; first find the hundreds, then the tens, then the ones.

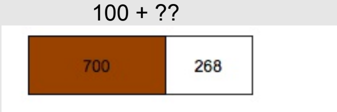
Method A



Method B



100 + ??

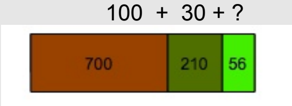


The length has 1 hundred, making a rectangle with area 700.

Method A records the difference of the areas as 966 – 700 = 266, showing the remaining area (266). Only hundreds are subtracted; the tens and ones digits do not change.

Method B records only the hundreds digit (2) of the difference and "brings down" the unchanged tens digit (6). These digits represent: 2 hundreds + 6 tens = 26 tens.

100 + 30 + ?

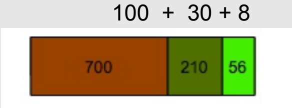


The length has 3 tens, making a rectangle with area 210.

Method A records the difference of the areas as 266 – 210 = 56. Only hundreds and tens are subtracted; the ones digit does not change.

Method B records only the tens digit (5) of the difference and "brings down" the ones digit (6). These digits represent: 5 tens + 6 ones = 56 ones.

100 + 30 + 8



The length has 8 ones, making an area of 56. The original rectangle can now be seen as composed of three smaller rectangles with areas of the amounts that were subtracted from 966.

Method A shows each partial quotient and has the final step adding them (going from 100 + 30 + 8 to 138).

Method B abbreviates these partial quotients. These can be said explicitly when explaining the method (e.g., 7 hundreds subtracted from the 9 hundreds is 2 hundreds).

966 ÷ 7 can be viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The divisor, partial quotients (100, 30, 8), and final quotient (138) represent quantities in length units and the dividend represents a quantity in area units.

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare, and how numbers round for decimals to thousandths.

New at Grade 5 is the use of whole number exponents to denote powers of 10.[5.NBT.2] Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right. So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole number and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value. Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

[5.NBT.2] Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Perform operations with multi-digit whole numbers and with decimals to hundredths At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

Division in Grade 5 extends Grade 4 methods to two-digit divisors.[5.NBT.6] Students continue to decompose the dividend into base-ten units and find the quotient place by place, starting from the highest place. They illustrate and explain their calculations using equations, rectangular arrays, and/or area models. Estimating the quotients is a new aspect of dividing by a two-digit number. Even if students round the dividend appropriately, the resulting estimate may need to be adjusted up or down. Sometimes multiplying the ones of a two-digit divisor composes a new thousand, hundred, or ten. These newly composed units can be written as part of the division computation, added mentally, or as part of a separate multiplication computation. Students who need to write decomposed units when subtracting need to remember to leave space to do so.

[5.NBT.5] Fluently multiply multi-digit whole numbers using the standard algorithm.

[5.NBT.6] Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Recording division after an underestimate

1655 ÷ 27	1
	10 ⎫ 61
Rounding 27	(30) 50
to 30 produces	27) 1655
the underestimate	-1350
50 at the first step	305
but this method	-270
allows the division	35
process to be	-27
continued	8

Because of the uniformity of the structure of the base-ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.[5.NBT.7] Like base-ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. A whole number is not usually written with a decimal point, but a decimal point followed by one or more 0s can be inserted on the right (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base-ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing newly composed units on the addition line, and they can decompose units wherever needed before subtracting.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single-digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base-ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.[5.NF.3] When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.[5.NF.4] Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation-based method is not reliable in all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place

[5.NBT.7] Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths to make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns, then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

[5.NF.7b] Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Extending beyond Grade 5

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. Students use these ideas again when they calculate with polynomials in high school.

The distributive property and like units: Multiplication of whole numbers and polynomials

$52 \cdot 73$	$(5x+2)(7x+3)$	
$= (5 \cdot 10 + 2)(7 \cdot 10 + 3)$	$= (5x+2)(7x+3)$	decomposing as like units (powers of 10 or powers of x)
$= 5 \cdot 10(7 \cdot 10 + 3) + 2 \cdot (7 \cdot 10 + 3)$	$= 5x(7x+3) + 2(7x+3)$	using the distributive property
$= 35 \cdot 10^2 + 15 \cdot 10 + 14 \cdot 10 + 2 \cdot 3$	$= 35x^2 + 15x + 14x + 2 \cdot 3$	using the distributive property again
$= 35 \cdot 10^2 + 29 \cdot 10 + 6$	$= 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)

Progressions for the Common Core State Standards in Mathematics (draft)

20 June 2011

K–3, Categorical Data; Grades 2–5, Measurement Data*

Overview

As students work with data in Grades K–5, they build foundations for their study of statistics and probability in Grades 6 and beyond, and they strengthen and apply what they are learning in arithmetic. Kindergarten work with data uses counting and order relations. First- and second-graders solve addition and subtraction problems in a data context. In Grades 3–5, work with data is closely related to the number line, fraction concepts, fraction arithmetic, and solving problems that involve the four operations. See Table 1 for these and other notable connections between arithmetic and data work in Grades K–5.

As shown in Table 1, the K–5 data standards run along two paths. One path deals with *categorical data* and focuses on bar graphs as a way to represent and analyze such data. Categorical data comes from sorting objects into categories—for example, sorting a jumble of alphabet blocks to form two stacks, a stack for vowels and a stack for consonants. In this case there are two categories (Vowels and Consonants). Students' work with categorical data in early grades will support their later work with bivariate categorical data and two-way tables in eighth grade (this is discussed further at the end of the Categorical Data Progression).

The other path deals with *measurement data*. As the name suggests, measurement data comes from taking measurements. For example, if every child in a class measures the length of his or her hand to the nearest centimeter, then a set of measurement data is obtained. Other ways to generate measurement data might include measuring liquid volumes with graduated cylinders or measuring room temperatures with a thermometer. In each case, the Standards call for students to represent measurement data with a *line plot*.

*These progressions concern Measurement and Data standards related to data. Other MD standards are discussed in the Geometric Measurement Progression.

This is a type of display that positions the data along the appropriate scale, drawn as a number line diagram. These plots have two names in common use, "dot plot" (because each observation is represented as a dot) and "line plot" (because each observation is represented above a number line diagram).

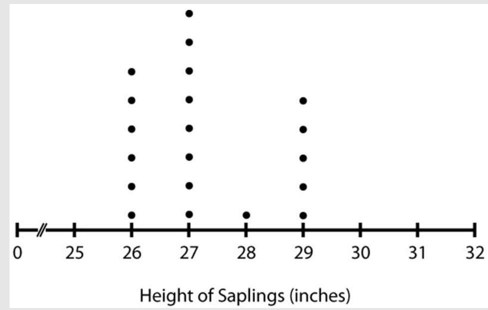
The number line diagram in a line plot corresponds to the scale on the measurement tool used to generate the data. In a context involving measurement of liquid volumes, the scale on a line plot could correspond to the scale etched on a graduated cylinder. In a context involving measurement of temperature, one might imagine a picture in which the scale on the line plot corresponds to the scale printed on a thermometer. In the last two cases, the correspondence may be more obvious when the scale on the line plot is drawn vertically.

Students should understand that the numbers on the scale of a line plot indicate the total number of measurement units from the zero of the scale.

Students need to choose appropriate representations (MP5), labeling axes to clarify the correspondence with the quantities in the situation and specifying units of measure (MP6). Measuring and recording data require attention to precision (MP6). Students should be supported as they learn to construct picture graphs, bar graphs, and line plots. Grid paper should be used for assignments as well as assessments. This may help to minimize errors arising from the need to track across a graph visually to identify values. Also, a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on. It might also help if students write relevant numbers on graphs during problem solving.

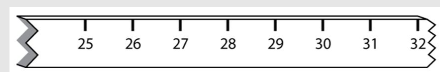
In students' work with data, context is important. As the *Guidelines for Assessment and Instruction in Statistics Education Report* notes, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning."• In keeping with this perspective, students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the data they represent (MP2).

Example of a line plot



Note that the break in the scale between 0 and 25 indicates that marks between 0 and 25 are not shown.

Example of a scale on a measurement tool



• The *Guidelines for Assessment and Instruction in Statistics Education Report* was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

Table 1: Some notable connections to K–5 data work

Grade	Standard	Notable Connections
	Categorical data	
K	K.MD.3. Classify objects into given categories, count the number of objects in each category and sort[1] the categories by count. *Limit category counts to be less than or equal to 10.*	• K.CC. Counting to tell the number of objects • K.CC. Comparing numbers
1	1.MD.4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.	• 1.OA. Problems involving addition and subtraction ◦ put-together, take-apart, compare ◦ problems that call for addition of three whole numbers
2	2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.	• 2.OA. Problems involving addition and subtraction ◦ put-together, take-apart, compare
3	3.MD.3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*	• 3.OA.3. Problems involving multiplication • 3.OA.8 Two-step problems using the four operations • 3.G.1 Categories of shapes
	Measurement data	
2	2.MD.9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.	• 1.MD.2. Length measurement • 2.MD.6. Number line
3	3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units–whole numbers, halves, or quarters.	• 3.NF.2. Fractions on a number line
4	4.MD.4. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*	• 4.NF.3,4. Problems involving fraction arithmetic
5	5.MD.2. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*	• 5.NF.1,2,4,6,7. Problems involving fraction arithmetic

[1] Here, "sort the categories" means "order the categories," i.e., show the categories in order according to their respective counts.

Categorical Data

Kindergarten

Students in Kindergarten classify objects into categories, initially specified by the teacher and perhaps eventually elicited from students. For example, in a science context, the teacher might ask students in the class to sort pictures of various organisms into two piles: organisms with wings and those without wings. Students can then count the number of specimens in each pile.[K.CC.5] Students can use these category counts and their understanding of cardinality to say whether there are more specimens with wings or without wings.[K.CC.6,K.CC.7]

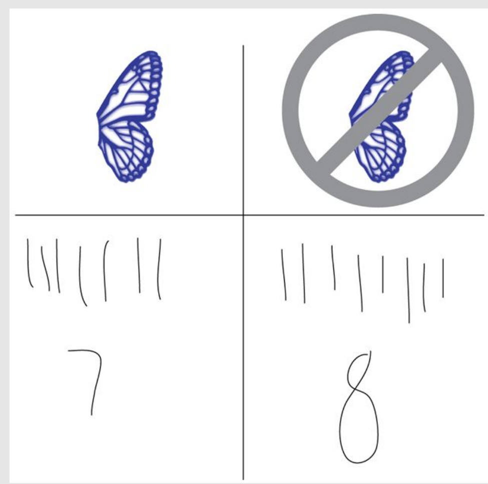
A single group of specimens might be classified in different ways, depending on which attribute has been identified as the attribute of interest. For example, some specimens might be insects, while others are not insects. Some specimens might live on land, while others live in water.

[K.CC.5] Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

[K.CC.6] Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

[K.CC.7] Compare two numbers between 1 and 10 presented as written numerals.

Sorting categorical data



The marks represent individual data points. The two category counts, 7 and 8, are a numerical summary *of the data.*

Grade 1

Students in Grade 1 begin to organize and represent categorical data. For example, if a collection of specimens is sorted into two piles based on which specimens have wings and which do not, students might represent the two piles of specimens on a piece of paper, by making a group of marks for each pile, as shown below (the marks could also be circles, for example). The groups of marks should be clearly labeled to reflect the attribute in question.

The work shown in the figure is the result of an intricate process. At first, we have before us a jumble of specimens with many attributes. Then there is a narrowing of attention to a single attribute (wings or not). Then the objects might be arranged into piles. The arranging of objects into piles is then mirrored in the arranging of marks into groups. In the end, each mark represents an object; its position in one column or the other indicates whether or not that object has a given attribute.

There is no single correct way to represent categorical data-and the Standards do not require Grade 1 students to use any specific format. However, students should be familiar with mark schemes like the one shown in the figure. Another format that might be useful in Grade 1 is a picture graph in which one picture represents one object. (Note that picture graphs are not an expectation in the Standards until Grade 2.) If different students devise different ways to represent the same data set, then the class might discuss relative strengths and weaknesses of each scheme (MP5).

Students' data work in Grade 1 has important connections to addition and subtraction, as noted in Table 1. Students in Grade 1 can ask and answer questions about categorical data based on a representation of the data. For example, with reference to the figure above, a student might ask how many specimens there were altogether, representing this problem by writing an equation such as $7 + 8 = \square$. Students can also ask and answer questions leading to other kinds of addition and subtraction problems (1.OA), such as compare problems or problems involving the addition of three numbers (for situations with three categories).

Grade 2

Students in Grade 2 draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. They solve simple put-together, take-apart, and compare problems using information presented in a bar graph.[2.MD.10, 2.OA.1]

The illustration shows an activity in which students make a bar graph to represent categorical data, then solve addition and subtraction problems based on the data. Students might use scissors to cut out the pictures of each organism and then sort the organisms into piles by category. Category counts might be recorded efficiently in the form of a table.

A bar graph representing categorical data displays no additional information beyond the category counts. In such a graph, the bars are a way to make the category counts easy to interpret visually. Thus, the word problem in part 4 could be solved without drawing a bar graph, just by using the category counts. The problem could even be cast entirely in words, without the accompanying picture: "There are 9 insects, 4 spiders, 13 vertebrates, and 2 organisms of other kinds. How many more spiders would there have to be in order for the number of spiders to equal the number of vertebrates?" Of course, in solving this problem, students would not need to participate in categorizing data or representing it.

Scales in bar graphs Consider the two bar graphs shown to the right, in which the bars are oriented vertically. (Bars in a bar graph can also be oriented horizontally, in which case the following discussion would be modified in the obvious way.) Both of these bar graphs represent the same data set.

These examples illustrate that the horizontal axis in a bar graph of categorical data is not a scale of any kind; position along the horizontal axis has no numerical meaning. Thus, the horizontal position and ordering of the bars are not determined by the data.•

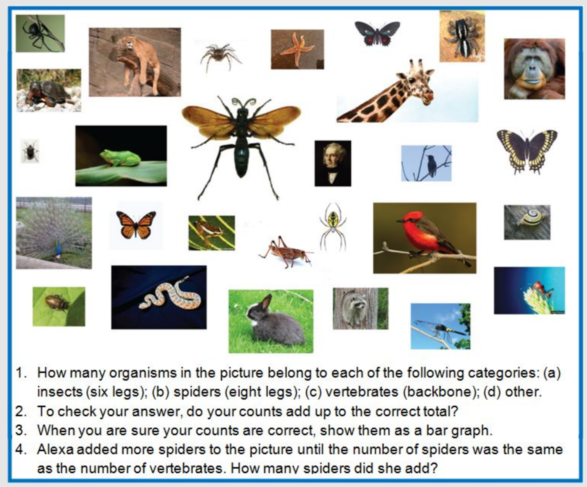
However, the vertical axes in these graphs do have numerical meaning. In fact, the vertical axes in these graphs are segments of number line diagrams. We might think of the vertical axis as a "count scale" (a scale showing counts in whole numbers)—as opposed to a measurement scale, which can be subdivided into fractions of a measurement unit.

Because the count scale in a bar graph is a segment of a number line diagram, when we answer a question such as "How many

2.MD.10 Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

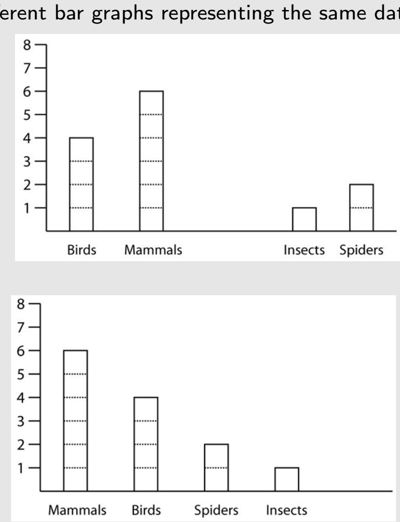
2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Activity for representing categorical data



Students might reflect on the way in which the category counts in part 1 of the activity enable them to efficiently solve the word problem in part 4. (The word problem in part 4 would be difficult to solve directly using just the array of images.)

Different bar graphs representing the same data set



• To minimize potential confusion, it might help to avoid presenting students with examples of categorical data in which the categories are named using numerals, e.g., "Candidate 1," "Candidate 2," "Candidate 3." This will ensure that the only numbers present in the display are found along the count scale.

more birds are there than spiders?" we are finding differences on a number line diagram.[2.MD.6]

When drawing bar graphs on grid paper, the tick marks on the count scale should be drawn at intersections of the gridlines. The tops of the bars should reach the respective gridlines of the appropriate tick marks. When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper.

Students could discuss ways in which bar orientation (horizontal or vertical), order, thickness, spacing, shading, colors, and so forth make the bar graphs easier or more difficult to interpret. By middle school, students could make thoughtful design choices about data displays, rather than just accepting the defaults in a software program (MP5).

Grade 3

In Grade 3, the most important development in data representation for categorical data is that students now draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor in order to yield the number of objects in the given category. These developments connect with the emphasis on multiplication in this grade.

At the end of Grade 3, students can draw a scaled picture graph or a scaled bar graph to represent a data set with several categories (six or fewer categories).[3.MD.3] They can solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.[3.OA.3,3.OA.8] See the examples in the margin, one of which involves categories of shapes.[3.G.1] As in Grade 2, category counts might be recorded efficiently in the form of a table.

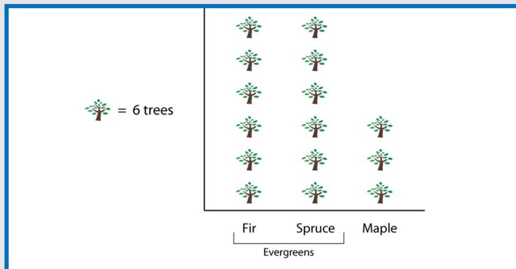
Students can gather categorical data in authentic contexts, including contexts arising in their study of science, history, health, and so on. Of course, students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets. The Standards in Grades 1–3 do not require students to gather categorical data.

Where the Categorical Data Progression is heading

Students' work with categorical data in early grades will develop into later work with bivariate categorical data and two-way tables in eighth grade. "Bivariate categorical data" are data that are categorized according to two attributes. For example, if there is an outbreak of stomach illness on a cruise ship, then passengers might be sorted in two different ways: by determining who got sick and

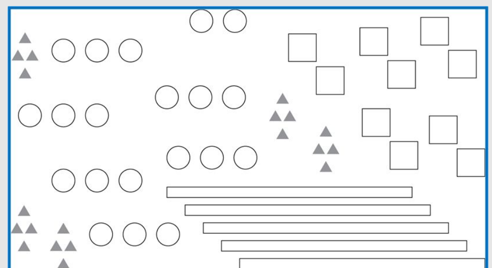
[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A problem about interpreting a scaled picture graph



The picture graph shows how many trees of each kind were in the arboretum. How many more evergreen trees than maple trees are there? (Fir trees and spruce trees are evergreen trees.)

Problems involving categorical data



1. How many shapes above belong to each of the following categories: (a) rectangles; (b) circles; (c) triangles.
2. When you are sure your counts are correct, show them as a picture graph. Use a scale with 1 picture = 5 shapes.
3. Alex added circles to the picture until the number of circles was the same as the number of polygons. How many circles did Alex add?

[3.MD.3] Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.3] Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities,. . .

[3.OA.8] Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity . . .

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals . . .

who didn't, and by determining who ate the shellfish and who didn't. This double categorization—normally shown in the form of a two-way table—might show a strong positive or negative association, in which case it might used to support or contest (but not prove or disprove) a claim about whether the shellfish was the cause of the illness.[8.SP.4]

[8.SP.4]Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.

Measurement Data

Grade 2

Students in Grade 2 measure lengths to generate a set of measurement data.[2.MD.1] For example, each student might measure the length of his or her arm in centimeters, or every student might measure the height of a statue in inches. (Students might also generate their own ideas about what to measure.) The resulting data set will be a list of observations, for example as shown in the margin on the following page for the scenario of 28 students each measuring the height of a statue. (This is a larger data set than students would normally be expected to work with in elementary grades.)

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

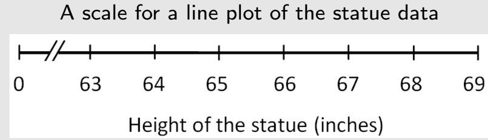
How might one summarize this data set or display it visually? Because students in Grade 2 are already familiar with categorical data and bar graphs, a student might find it natural to summarize this data set by viewing it in terms of categories—the categories in question being the six distinct height values which appear in the data (63 inches, 64 inches, 65 inches, 66 inches, 67 inches, and 69 inches). For example, the student might want to say that there are four observations in the "category" of 67 inches. However, it is important to recognize that 64 inches is not a category like "spiders." Unlike "spiders," 63 inches is a numerical value with a measurement unit. That difference is why the data in this table are called measurement data.

A display of measurement data must present the measured values with their appropriate magnitudes and spacing on the measurement scale in question (length, temperature, liquid capacity, etc.). One method for doing this is to make a *line plot*. This activity connects with other work students are doing in measurement in Grade 2: representing whole numbers on number line diagrams, and representing sums and differences on such diagrams.[2.MD.5,2.MD.6]

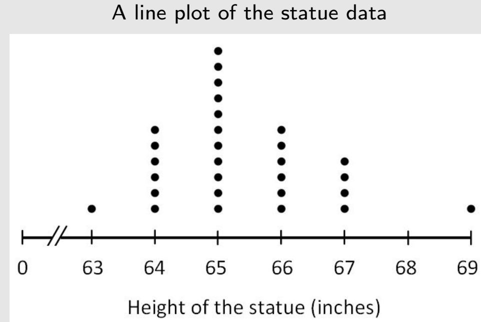
2.MD.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

A scale for a line plot of the statue data



A line plot of the statue data



To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: 63 inches and 69 inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale.

Note that the value 68 inches, which was not present in the data set, has been written in proper position midway between 67 inches and 69 inches. (This need to fill in gaps does not exist for a categorical data set; there no "gap" between categories such as fish and spiders!)

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. If a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the observations, or to do any counting of them, before producing the line plot. (In fact, one could even assemble the line plot as the data are being collected, at the expense of having a record of who made what measurement. Students might discuss whether such a record is valuable and why.)

Students might enjoy discussing and interpreting visual features of line plots, such as the "outlier" value of 69 inches in this line plot. (Did student #13 make a serious error in measuring the statue's height? Or in fact is student #13 the only person in the class who measured the height correctly?) However, in Grade 2 the only requirement of the Standards dealing with measurement data is that students generate measurement data and build line plots to display the resulting data sets. (Students do not have to generate the data every time they work on making line plots. That would be too time-consuming. After some experiences in generating the data, most work in producing line plots can be done by providing students with data sets.)

Grid paper might not be as useful for drawing line plots as it is for bar graphs, because the count scale on a line plot is seldom shown for the small data sets encountered in the elementary grades. Additionally, grid paper is usually based on a square grid, but the count scale and the measurement scale of a line plot are conceptually distinct, and there is no need for the measurement unit on the measurement scale to be drawn the same size as the counting unit on the count scale.

Grade 3

In Grade 3, students are beginning to learn fraction concepts (3.NF). They understand fraction equivalence in simple cases, and they use visual fraction models to represent and order fractions. Grade 3 students also measure lengths using rulers marked with halves and fourths of an inch. They use their developing knowledge of fractions and number lines to extend their work from the previous grade by working with measurement data involving fractional measurement values.

For example, every student in the class might measure the height of a bamboo shoot growing in the classroom, leading to the data set shown in the table. (Again, this illustration shows a larger data set than students would normally work with in elementary grades.)

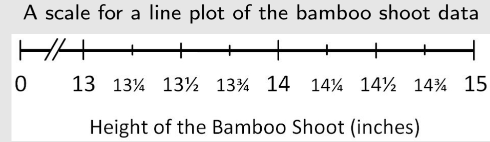
To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: $13\frac{1}{2}$ inches and $14\frac{3}{4}$ inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale. This is just like part of the scale on a ruler.

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. As with Grade 2 line plots, if a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the

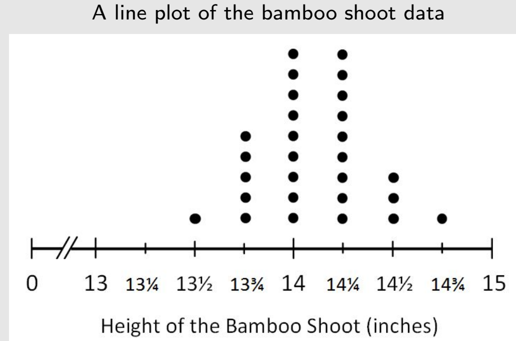
Students' measurements of a statue and of a bamboo shoot

Statue measurements: Student's initials	Statue measurements: Student's measured value (inches)	Bamboo shoot measurements: Student's initials	Bamboo shoot measurements: Height value (inches)
W.B.	64	W.B.	13 3/4
D.W.	65	D.W.	14 1/2
H.D.	65	H.D.	14 1/4
G.W.	65	G.W.	14 3/4
V.Y.	67	V.Y.	14 1/4
T.T.	66	T.T.	14 1/2
D.F.	67	D.F.	14
B.H.	65	B.H.	13 1/2
H.H.	63	H.H.	14 1/4
V.H.	64	V.H.	14 1/4
I.O.	64	I.O.	14 1/4
W.N.	65	W.N.	14
B.P.	69	B.P.	14 1/2
V.A.	65	V.A.	13 3/4
H.L.	66	H.L.	14
O.M.	64	O.M.	13 3/4
L.E.	65	L.E.	14 1/4
M.J.	66	M.J.	13 3/4
T.D.	66	T.D.	14 1/4
K.P.	64	K.P.	14
H.N.	65	H.N.	14
W.M.	67	W.M.	14
C.Z.	64	C.Z.	13 3/4
J.I.	66	J.I.	14
M.S.	66	M.S.	14 1/4
T.C.	65	T.C.	14
G.V.	67	G.V.	14
O.F.	65	O.F.	14 1/4

A scale for a line plot of the bamboo shoot data



A line plot of the bamboo shoot data



observations, or to do any counting of them, before producing the line plot.

Students can pose questions about data presented in line plots, such as how many students obtained measurements larger than $14\frac{1}{4}$ inches.

Grades 4 and 5

Grade 4 students learn elements of fraction equivalence[4.NF.1] and arithmetic, including multiplying a fraction by a whole number[4.NF.4] and adding and subtracting fractions with like denominators.[4.NF.3] Students can use these skills to solve problems, including problems that arise from analyzing line plots. For example, with reference to the line plot above, students might find the difference between the greatest and least values in the data. (In solving such problems, students may need to label the measurement scale in eighths so as to produce like denominators. Decimal data can also be used in this grade.)

Grade 5 students grow in their skill and understanding of fraction arithmetic, including multiplying a fraction by a fraction,[5.NF.4] dividing a unit fraction by a whole number or a whole number by a unit fraction,[4.NF.7] and adding and subtracting fractions with unlike denominators.[5.NF.1] Students can use these skills to solve problems,[5.NF.2,5.NF.6,5.NF.7c] including problems that arise from analyzing line plots. For example, given five graduated cylinders with different measures of liquid in each, students might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. (Students in Grade 6 will view the answer to this question as the mean value for the data set in question.)

As in earlier grades, students should work with data in science and other subjects. Grade 5 students working in these contexts should be able to give deeper interpretations of data than in earlier grades, such as interpretations that involve informal recognition of pronounced differences in populations. This prefigures the work they will do in middle school involving distributions, comparisons of populations, and inference.

Where the Measurement Data Progression is heading

Connection to Statistics and Probability By the end of Grade 5, students should be comfortable making line plots for measurement data and analyzing data shown in the form of a line plot. In Grade 6, students will take an important step toward statistical reasoning per se when they approach line plots as pictures of distributions with features such as clustering and outliers.

Students' work with line plots during the elementary grades develops in two distinct ways during middle school. The first development comes in sixth grade,[6.SP.4] when *histograms* are used.[1] Like

[4.NF.1] Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

[4.NF.3] Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.

[5.NF.1] Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

[5.NF.2] Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

[5.NF.6] Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.7c] Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

[6.SP.4] Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

line plots, histograms have a measurement scale and a count scale; thus, a histogram is a natural evolution of a line plot and is used for similar kinds of data (univariate measurement data, the kind of data discussed above).

The other evolution of line plots in middle school is arguably more important. It involves the graphing of bivariate measurement data.[8.SP.1-3] "Bivariate measurement data" are data that represent two measurements. For example, if you take a temperature reading every ten minutes, then every data point is a measurement of temperature as well as a measurement of time. Representing two measurements requires two measurement scales—or in other words, a coordinate plane in which the two axes are each marked in the relevant measurement units. Representations of bivariate measurement data in the coordinate plane are called *scatter plots*. In the case where one axis is a time scale, they are called *time graphs* or *line graphs*. Time graphs can be used to visualize trends over time, and scatter plots can be used to discover associations between measured variables in general.

8.SP.1 Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

8.SP.2 Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

8.SP.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

Connection to the Number System The Standards do not explicitly require students to create time graphs. However, it might be considered valuable to expose students to time series data and to time graphs as part of their work in meeting standard 6.NS.8. For example, students could create time graphs of temperature measured each hour over a 24-hour period, where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day. It is traditional to connect ordered pairs with line segments in such a graph, not in order to make any claims about the actual temperature value at unmeasured times, but simply to aid the eye in perceiving trends.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

[1] To display a set of measurement data with a histogram, specify a set of nonoverlapping intervals along the measurement scale. Then, instead of showing each individual measurement as a dot, use a bar oriented along the count scale to indicate the number of measurements lying within each interval on the measurement scale. A histogram is thus a little like a bar graph for categorical data, except that the "categories" are successive intervals along a measurement scale. (Note that the Standards follow the GAISE report in reserving the term "categorical data" for non-numerical categories. In the Standards, as in GAISE (see p. 35), bar graphs are for categorical data with non-numerical categories, while histograms are for measurement data which has been grouped by intervals along the measurement scale.)

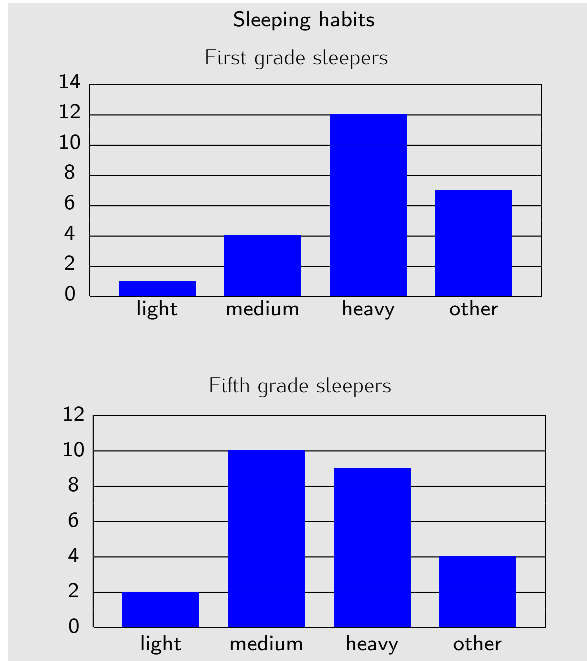
Appendix: Additional Examples

These examples show some rich possibilities for data work in K–8. The examples are not shown by grade level because each includes some aspects that go beyond the expectations stated in the Standards.

Example 1. Comparing bar graphs

Are younger students lighter sleepers than older students? To study this question a class first agreed on definitions for light, medium and heavy sleepers and then collected data from first and fifth grade students on their sleeping habits. The results are shown in the margin.

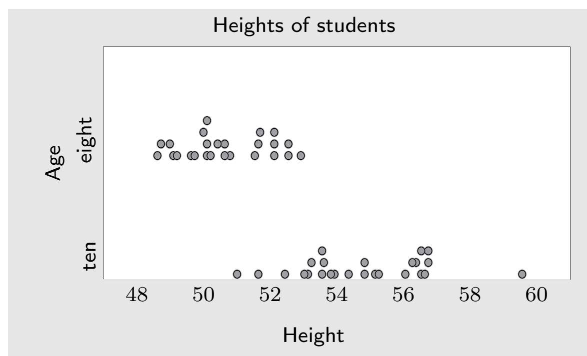
How do the patterns differ? What is the typical value for first graders? What is the typical value for fifth graders? Which of these groups appears to be the heavier sleepers?



Example 2. Comparing line plots

Fourth grade students interested in seeing how heights of students change for kids around their age measured the heights of a sample of eight-year-olds and a sample of ten-year-olds. Their data are plotted in the margin.

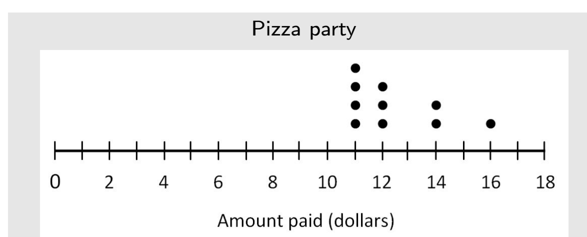
Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?



Example 3. Fair share averaging

Ten students decide to have a pizza party and each is asked to bring his or her favorite pizza. The amount paid (in dollars) for each pizza is shown in the plot to the right.

Each of the ten is asked to contribute an equal amount (his or her fair share) to the cost of the pizza. Where does that fair share amount lie on the plot? Is it closer to the smaller values or the large one? Now, two more students show up for the party and they have contributed no pizza. Plot their values on the graph and calculate a new fair share. Where does it lie on the plot? How many more students without pizza would have to show up to bring the fair share cost below \$8.00?



Progressions for the Common Core State Standards in Mathematics (draft)

23 June 2012

K–5, Geometric Measurement

1

Overview

Geometric measurement connects the two most critical domains of early mathematics, geometry and number, with each providing conceptual support to the other. Measurement is central to mathematics, to other areas of mathematics (e.g., laying a sensory and conceptual foundation for arithmetic with fractions), to other subject matter domains, especially science, and to activities in everyday life. For these reasons, measurement is a core component of the mathematics curriculum.

Measurement is the process of assigning a number to a magnitude of some attribute shared by some class of objects, such as length, relative to a unit. Length is a *continuous* attribute—a length can always be subdivided in smaller lengths. In contrast, we can count 4 apples exactly—cardinality is a discrete attribute. We can add the 4 apples to 5 other apples and know that the result is exactly 9 apples. However, the *weight* of those apples is a continuous attribute, and scientific measurement with tools gives only an approximate measurement—to the nearest pound (or, better, kilogram) or the nearest 1/100$^{\text{th}}$ of a pound, but always with some error.•

• The Standards do not differentiate between weight and mass. Technically, mass is the amount of matter in an object. Weight is the force exerted on the body by gravity. On the earth's surface, the distinction is not important (on the moon, an object would have the same mass, would weight less due to the lower gravity).

Before learning to measure attributes, children need to recognize them, distinguishing them from other attributes. That is, the attribute to be measured has to "stand out" for the student and be discriminated from the undifferentiated sense of amount that young children often have, labeling greater lengths, areas, volumes, and so forth, as "big" or "bigger."

Students then can become increasingly competent at *direct comparison*—comparing the amount of an attribute in two objects without measurement. For example, two students may stand back to back to directly compare their heights. In many circumstances, such direct comparison is impossible or unwieldy. Sometimes, a third object can be used as an intermediary, allowing *indirect comparison*. For example, if we know that Aleisha is taller than Barbara and that

[1] This progression concerns Measurement and Data standards related to geometric measurement. The remaining Measurement and Data standards are discussed in the K–3 Categorical Data and Grades 2–5 Measurement Data Progressions.

Barbara is taller than Callie, then we know (due to the transitivity of "taller than") that Aleisha is taller than Callie, even if Aleisha and Callie never stand back to back.•

• "Transitivity" abbreviates the Transitivity Principle for Indirect Measurement stated in the Standards as:

> If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

The purpose of measurement is to allow indirect comparisons of objects' amount of an attribute using numbers. An attribute of an object is measured (i.e., assigned a number) by comparing it to an amount of that attribute held by another object. One measures length with length, mass with mass, torque with torque, and so on. In geometric measurement, a unit is chosen and the object is subdivided or partitioned by copies of that unit and, to the necessary degree of precision, units subordinate to the chosen unit, to determine the number of units and subordinate units in the partition.

Personal benchmarks, such as "tall as a doorway" build students' intuitions for amounts of a quantity and help them use measurements to solve practical problems. A combination of internalized units and measurement processes allows students to develop increasing accurate estimation competencies.

Both in measurement and in estimation, the concept of *unit* is crucial. The concept of basic (as opposed to subordinate) unit just discussed is one aspect of this concept. The basic unit can be informal (e.g., about a car length) or standard (e.g., a meter). The distinction and relationship between the notion of discrete "1" (e.g., one apple) and the continuous "1" (e.g., one inch) is important mathematically and is important in understanding number line diagrams (e.g., see Grade 2) and fractions (e.g., see Grade 3). However, there are also superordinate units or "units of units." A simple example is a kilometer consisting of 1,000 meters. Of course, this parallels the number concepts students must learn, as understanding that tens and hundreds are, respectively, "units of units" and "units of units of units" (i.e., students should learn that 100 can be simultaneously considered as 1 hundred, 10 tens, and 100 ones).

Students' understanding of an attribute that is measured with derived units is dependent upon their understanding that attribute as entailing other attributes simultaneously. For example,

- Area as entailing two lengths, simultaneously;
- Volume as entailing area and length (and thereby three lengths), simultaneously.

Scientists measure many types of attributes, from hardness of minerals to speed. This progression emphasizes the geometric attributes of length, area, and volume. Nongeometric attributes such as weight, mass, capacity, time, and color, are often taught effectively in science and social studies curricula and thus are not extensively discussed here. Attributes derived from two different attributes, such as speed (derived from distance and time), are discussed in the high school Number and Quantity Progression and in the 6–7 Ratio and Proportion Progression.

Length is a characteristic of an object found by quantifying how far it is between the endpoints of the object. "Distance" is often used similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects, choosing a unit of measure and *subdividing* (mentally and physically) the object by that unit, placing that unit end to end (*iterating*) alongside the object. The length of the object is the number of units required to iterate from one end of the object to the other, without gaps or overlaps.

Length is a core concept for several reasons. It is the basic geometric measurement. It is also involved in area and volume measurement, especially once formulas are used. Length and unit iteration are critical in understanding and using the number line in Grade 3 and beyond (see the Number and Operations—Fractions Progression). Length is also one of the most prevalent metaphors for quantity and number, e.g., as the master metaphor for magnitude (e.g., vectors, see the Number and Quantity Progression). Thus, length plays a special role in this progression.

Area is an amount of two-dimensional surface that is contained within a plane figure. Area measurement assumes that congruent figures enclose equal areas, and that area is *additive*, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas. Area is measured by tiling a region with a two-dimensional unit (such as a square) and parts of the unit, without gaps or overlaps. Understanding how to spatially structure a two-dimensional region is an important aspect of the progression in learning about area.

Volume is an amount of three-dimensional space that is contained within a three-dimensional shape. Volume measurement assumes that congruent shapes enclose equal volumes, and that volume is *additive*, i.e., the volume of the union of two regions that overlap only at their boundaries is the sum of their volumes. Volume is measured by packing (or tiling, or tessellating) a region with a three-dimensional unit (such as a cube) and parts of the unit, without gaps or overlaps. Volume not only introduces a third dimension and thus an even more challenging spatial structuring, but also complexity in the nature of the materials measured. That is, solid units might be "packed," such as cubes in a three-dimensional array or cubic meters of coal, whereas liquids "fill" three-dimensional regions, taking the shape of a container, and are often measured in units such as liters or quarts.

A final, distinct, geometric attribute is *angle measure*. The size of an angle is the amount of rotation between the two rays that form the angle, sometimes called the sides of the angles.

Finally, although the attributes that we measure differ as just described, it is important to note: *central characteristics of measurement are the same for all of these attributes.* As one more testament to these similarities, consider the following side-by-side comparison of the Standards for measurement of area in Grade 3 and the measurement of volume in Grade 5.

Grade 3

Understand concepts of area and relate area to multiplication and to addition.

3.MD.5. Recognize area as an attribute of plane figures and understand concepts of area measurement.

a. A square with side length 1 unit, called “a unit square,” is said to have “one square unit” of area, and can be used to measure area.
b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7. Relate area to the operations of multiplication and addition.

a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Grade 5

Understand concepts of volume and relate volume to multiplication and to addition.

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

a. A cube with side length 1 unit, called a “unit cube,” is said to have “one cubic unit” of volume, and can be used to measure volume.
b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5.MD.5 Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.
c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Kindergarten

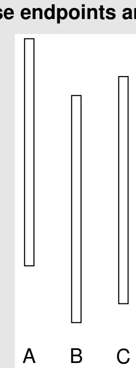
Describe and compare measurable attributes Students often initially hold undifferentiated views of measurable attributes, saying that one object is "bigger" than another whether it is longer, or greater in area, or greater in volume, and so forth. For example, two students might both claim their block building is "the biggest." Conversations about how they are comparing—one building may be taller (greater in length) and another may have a larger base (greater in area)—help students learn to discriminate and name these measureable attributes. As they discuss these situations and compare objects using different attributes, they learn to distinguish, label, and describe several measureable attributes of a single object.[K.MD.1] Thus, teachers listen for and extend conversations about things that are "big," or "small," as well as "long," "tall," or "high," and name, discuss, and demonstrate with gestures the attribute being discussed (length as extension in one dimension is most common, but area, volume, or even weight in others).

K.MD.1 Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

Length Of course, such conversations often occur in comparison situations ("He has more than me!"). Kindergartners easily directly compare lengths in simple situations, such as comparing people's heights, because standing next to each other automatically aligns one endpoint.[K.MD.2] However, in other situations they may initially compare only one endpoint of objects to say which is longer. Discussing such situations (e.g., when a child claims that he is "tallest" because he is standing on a chair) can help students resolve and coordinate perceptual and conceptual information when it conflicts. Teachers can reinforce these understandings, for example, by holding two pencils in their hand showing only one end of each, with the longer pencil protruding less. After asking if they can tell which pencil is longer, they reveal the pencils and discuss whether children were "fooled." The necessity of aligning endpoints can be explicitly addressed and then re-introduced in the many situations throughout the day that call for such comparisons. Students can also make such comparisons by moving shapes together to see which has a longer side.

K.MD.2 Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference.

Sticks whose endpoints are not aligned



When shown this figure and asked which is "the longest stick," students may point to A because it "sticks out the farthest." Similarly, they may recognize a 12-inch vertical line as "tall" and a 12-inch horizontal line as "long" but not recognize that the two are the same length.

Even when students seem to understand length in such activities, they may not conserve length. That is, they may believe that if one of two sticks of equal lengths is vertical, it is then longer than the other, horizontal, stick. Or, they may believe that a string, when bent or curved, is now shorter (due to its endpoints being closer to each other). Both informal and structured experiences, including demonstrations and discussions, can clarify how length is maintained, or conserved, in such situations. For example, teachers and students might rotate shapes to see its sides in different orientations. As with number, learning and using language such as "It looks longer, but it really isn't longer" is helpful.

Students who have these competencies can engage in experiences that lay the groundwork for later learning. Many can begin to learn to compare the lengths of two objects using a third object, order lengths, and connect number to length. For example, informal experiences such as making a road "10 blocks long" help students build a foundation for measuring length in the elementary grades. See the Grade 1 section on length for information about these important developments.

Area and volume Although area and volume experiences are not instructional foci for Kindergarten, they are attended to, at least to distinguish these attributes from length, as previously described. Further, certain common activities can help build students' experiential foundations for measurement in later grades. Understanding area requires understanding this attribute as the amount of two-dimensional space that is contained within a boundary. Kindergartners might informally notice and compare areas associated with everyday activities, such as laying two pieces of paper on top of each other to find out which will allow a "bigger drawing." Spatial structuring activities described in the Geometry Progression, in which designs are made with squares covering rectilinear shapes also help to create a foundation for understanding area.

Similarly, kindergartners might compare the capacities of containers informally by pouring (water, sand, etc.) from one to the other. They can try to find out which holds the most, recording that, for example, the container labeled "J" holds more than the container labeled "D" because when J was poured into D it overflowed. Finally, in play, kindergartners might make buildings that have layers of rectangular arrays. Teachers aware of the connections of such activities to later mathematics can support students' growth in multiple domains (e.g., development of self-regulation, social-emotional, spatial, and mathematics competencies) simultaneously, with each domain supporting the other.

Grade 1

Length comparisons First graders should continue to use direct comparison—carefully, considering all endpoints—when that is appropriate. In situations where direct comparison is not possible or convenient, they should be able to use indirect comparison and explanations that draw on transitivity (MP3). Once they can compare lengths of objects by direct comparison, they could compare several items to a single item, such as finding all the objects in the classroom the same length as (or longer than, or shorter than) their forearm.[1.MD.1] Ideas of transitivity can then be discussed as they use a string to represent their forear's length. As another example, students can figure out that one path from the teachers' desk to the door is longer than another because the first path is longer than a length of string laid along the path, but the other path is shorter than that string. Transitivity can then be explicitly discussed: If A is longer than B and B is longer than C, then A must be longer than C as well.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Seriation Another important set of skills and understandings is ordering a set of objects by length.[1.MD.1] Such sequencing requires multiple comparisons. Initially, students find it difficult to seriate a large set of objects (e.g., more than 6 objects) that differ only slightly in length. They tend to order groups of two or three objects, but they cannot correctly combine these groups while putting the objects in order. Completing this task efficiently requires a systematic strategy, such as moving each new object "down the line" to see where it fits. Students need to understand that each object in a seriation is larger than those that come before it, and shorter than those that come after. Again, reasoning that draws on transitivity is relevant.

1.MD.1 Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Such seriation and other processes associated with the measurement and data standards are important in themselves, but also play a fundamental role in students' development. The general reasoning processes of seriation, conservation (of length and number), and classification (which lies at the heart of the standards discussed in the K–3 Categorical Data Progression) predict success in early childhood as well as later schooling.

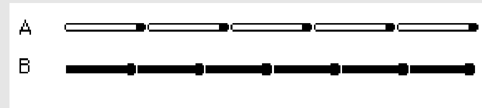
Measure lengths indirectly and by iterating length units Directly comparing objects, indirectly comparing objects, and ordering objects by length are important practically and mathematically, but they are not length measurement, which involves assigning a number to a length. Students learn to lay physical units such as centimeter or inch manipulatives end-to-end and count them to measure a length.[1.MD.2] Such a procedure may seem to adults to be straightforward, however, students may initially iterate a unit leaving gaps between subsequent units or overlapping adjacent units. For such students, measuring may be an activity of placing units along a

1.MD.2 Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

path in some manner, rather than the activity of covering a region or length with no gaps.

Also, students, especially if they lack explicit experience with continuous attributes, may make their initial measurement judgments based upon experiences counting discrete objects. For example, researchers showed children two rows of matches. The matches in each row were of different lengths, but there was a different number of matches in each so that the rows were the same length. Although, from the adult perspective, the lengths of the rows were the same, many children argued that the row with 6 matches was longer because it had more matches. They counted units (matches), assigning a number to a *discrete* attribute (cardinality). In measuring *continuous* attributes, the sizes of the units (white and dark matches) must be considered. First grade students can learn that objects used as basic units of measurement (e.g., "match-length") must be the same size.

Matches of different lengths



Row A is 5 matches long–when the unit of measurement is white matches. Row B is 6 matches long–when the unit of measurement is dark matches. From Inhelder, Sinclair, and Bovet, 1974, *Learning and the Development of Cognition*, *Harvard University Press.*

As with transitive reasoning tasks, using comparison tasks and asking children to compare results can help reveal the limitations of such procedures and promote more accurate measuring. However, students also need to see agreements. For example, understanding that the results of measurement and direct comparison have the same results encourages children to use measurement strategies.

Another important issue concerns the use of standard or nonstandard units of length. Many curricula or other instructional guides advise a sequence of instruction in which students compare lengths, measure with nonstandard units (e.g., paper clips), incorporate the use of manipulative standard units (e.g., inch cubes), and measure with a ruler. This approach is probably intended to help students see the need for standardization. However, the use of a variety of different length units, *before students understand the concepts, procedures, and usefulness of measurement*, may actually deter students' development. Instead, students might learn to measure correctly with standard units, and even learn to use rulers, before they can successfully use nonstandard units and understand relationships between different units of measurement. To realize that arbitrary (and especially mixed-size) units result in the same length being described by different numbers, a student must reconcile the varying lengths and numbers of arbitrary units. Emphasizing nonstandard units too early may defeat the purpose it is intended to achieve. Early use of many nonstandard units may actually interfere with students' development of basic measurement concepts required to understand the need for standard units. In contrast, using manipulative standard units, or even standard rulers, is less demanding and appears to be a more interesting and meaningful real-world activity for young students.

Thus, an instructional progression based on this finding would start by ensuring that students can perform direct comparisons. Then, children should engage in experiences that allow them to connect number to length, using manipulative units that have a standard unit of length, such as centimeter cubes. These can be labeled "length-units" with the students. Students learn to lay such physical units end-to-end and count them to measure a length. They compare the results of measuring to direct and indirect comparisons.

As they measure with these manipulative units, students discuss the concepts and skills involved (e.g., as previously discussed, not leaving space between successive length-units). As another example, students initially may not extend the unit past the endpoint of the object they are measuring. If students make procedural errors such as these, they can be asked to tell in a precise and elaborate manner *what* the problem is, *why* it leads to incorrect measurements, and *how* to fix it and measure accurately.

Measurement activities can also develop other areas of mathematics, including reasoning and logic. In one class, first graders were studying mathematics mainly through measurement, rather than counting discrete objects. They described and represented relationships among and between lengths (MP2, MP3), such as comparing two sticks and symbolizing the lengths as "$A < B$." This enabled them to reason about relationships. For example, after seeing the following statements recorded on the board, if $V > M$, then $M \neq V$, $V \neq M$, and $M < V$, one first-grader noted, "If it's an inequality, then you can write four statements. If it's equal, you can only write two"(MP8)

This indicates that with high-quality experiences (such as those described in the Grade 2 section on length), many first graders can also learn to use reasoning, connecting this to direct comparison, and to measurement performed by laying physical units end-to-end.

Area and volume: Foundations As in Kindergarten, area and volume are not instructional foci for first grade, but some everyday activities can form an experiential foundation for later instruction in these topics. For example, in later grades, understanding area requires seeing how to decompose shapes into parts and how to move and recombine the parts to make simpler shapes whose areas are already known (MP7). First graders learn the foundations of such procedures both in composing and decomposing shapes, discussed in the Geometry Progression, and in comparing areas in specific contexts. For example, paper-folding activities lend themselves not just to explorations of symmetry but also to equal-area congruent parts. Some students can compare the area of two pieces of paper by cutting and overlaying them. Such experiences provide only initial development of area concepts, but these key foundations are important for later learning.

Volume can involve liquids or solids. This leads to two ways to measure volume, illustrated by "packing" a space such as a three-dimensional array with cubic units and "filling" with iterations of a fluid unit that takes the shape of the container (called liquid volume). Many first graders initially perceive filling as having a one-dimensional unit structure. For example, students may simply "read off" the measure on a graduated cylinder. Thus, in a science or "free time" activity, students might compare the volume of two containers in at least two ways. They might pour each into a graduated cylinder to compare the measures. Or they might practice indirect comparison using transitive reasoning by using a third container to compare the volumes of the two containers. By packing unit cubes into containers into which cubes fit readily, students also can lay a foundation for later "packing" volume.

Grade 2

Measure and estimate lengths in standard units Second graders learn to measure length with a variety of tools, such as rulers, meter sticks, and measuring tapes.[2.MD.1] Although this appears to some adults to be relatively simple, there are many conceptual and procedural issues to address. For example, students may begin counting at the numeral "1" on a ruler. The numerals on a ruler may signify to students when to start counting, rather than the amount of space that has already been covered. It is vital that students learn that "one" represents the space from the beginning of the ruler to the hash mark, not the hash mark itself. Again, students may not understand that units must be of equal size. They will even measure with tools subdivided into units of different sizes and conclude that quantities with more units are larger.

To learn measurement concepts and skills, students might use both simple rulers (e.g., having only whole units such as centimeters or inches) and physical units (e.g., manipulatives that are centimeter or inch lengths). As described for Grade 1, teachers and students can call these "length-units." Initially, students lay multiple copies of the same physical unit end-to-end along the ruler. They can also progress to iterating with one physical unit (i.e., repeatedly marking off its endpoint, then moving it to the next position), even though this is more difficult physically and conceptually. To help them make the transition to this more sophisticated understanding of measurement, students might draw length unit marks along sides of geometric shapes or other lengths to see the unit lengths. As they measure with these tools, students with the help of the teacher discuss the concepts and skills involved, such as the following.

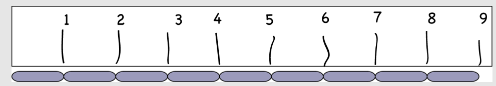
- *length-unit iteration*. E.g., not leaving space between successive length-units;
- *accumulation of distance*. Understanding that the counting "eight" when placing the last length-unit means the space covered by 8 length-units, rather then just the eighth length-unit (note the connection to cardinality[K.CC.4]);
- *alignment of zero-point*. Correct alignment of the zero-point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;
- *meaning of numerals on the ruler*. The numerals indicate the number of length units so far;
- *connecting measurement with physical units and with a ruler*. Measuring by laying physical units end-to-end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.

[2.MD.1] Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

[K.CC.4] Understand the relationship between numbers and quantities; connect counting to cardinality.

Students also can learn accurate procedures and concepts by drawing simple unit rulers. Using copies of a single length-unit such as inch-long manipulatives, they mark off length-units on strips of paper, explicitly connecting measurement with the ruler to measurement by iterating physical units. Thus, students' first rulers should be simply ways to help count the iteration of length-units. Frequently comparing results of measuring the same object with manipulative standard units and with these rulers helps students connect their experiences and ideas. As they build and use these tools, they develop the ideas of length-unit iteration, correct alignment (with a ruler), and the zero-point concept (the idea that the zero of the ruler indicates one endpoint of a length). These are reinforced as children compare the results of measuring to compare to objects with the results of directly comparing these objects.

Using a unit to draw a ruler



Students use a standard unit (shown in below the ruler) to make rulers, helping them to understand the meaning of the markings on rulers.

Similarly, discussions might frequently focus on "*What* are you counting?" with the answer being "length-units" or "centimeters" or the like. This is especially important because counting discrete items often convinces students that the size of things counted does not matter (there could be exactly 10 toys, even if they are different sizes). In contrast, for measurement, unit size is critical, so teachers are advised to plan experiences and reflections on the use of other units and length-units in various discrete counting and measurement contexts. Given that counting discrete items often correctly teaches students that the length-unit size does not matter, so teachers are advised to plan experiences and reflections on the use of units in various discrete counting and measurement contexts. For example, a teacher might challenge students to consider a fictitious student's measurement in which he lined up three large and four small blocks and claimed a path was "seven blocks long." Students can discuss whether he is correct or not.

Second graders also learn the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a specific length or distance.[2.MD.2] For example, it will take more centimeter lengths to cover a certain distance than inch lengths because inches are the larger unit. Initially, students may not appreciate the need for identical units. Previously described work with manipulative units of standard measure (e.g., 1 inch or 1 cm), along with related use of rulers and consistent discussion, will help children learn both the concepts and procedures of linear measurement. Thus, second grade students can learn that the larger the unit, the fewer number of units in a given measurement (as was illustrated on p. 9). That is, for measurements of a given length there is an inverse relationship between the size of the unit of measure and the number of those units. This is the time that measuring and reflecting on measuring the same object with different units, both standard and nonstandard, is likely to be most productive (see the discussion of this issue in the Grade 1 section on length). Results of measuring with different nonstandard length-units can be explicitly compared. Students also can use the concept of unit to make

[2.MD.2] Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

inferences about the relative sizes of objects; for example, if object A is 10 regular paperclips long and object B is 10 jumbo paperclips long, the number of units is the same, but the units have different sizes, so the lengths of A and B are different.

Second graders also learn to combine and compare lengths using arithmetic operations. That is, they can add two lengths to obtain the length of the whole and subtract one length from another to find out the difference in lengths.[2.MD.4] For example, they can use a simple unit ruler or put a length of connecting cubes together to measure first one modeling clay "snake," then another, to find the total of their lengths. The snakes can be laid along a line, allowing students to compare the measurement of that length with the sum of the two measurements. Second graders also begin to apply the concept of length in less obvious cases, such as the width of a circle, the length and width of a rectangle, the diagonal of a quadrilateral, or the height of a pyramid. As an arithmetic example, students might measure all the sides of a table with unmarked (foot) rulers to measure how much ribbon they would need to decorate the perimeter of the table.[2.MD.5] They learn to measure two objects and subtract the smaller measurement from the larger to find how much longer one object is than the other.

Second graders can also learn to represent and solve numerical problems about length using tape or number-bond diagrams. (See p. 16 of the Operations and Algebraic Thinking Progression for discussion of when and how these diagrams are used in Grade 1.) Students might solve two-step numerical problems at different levels of sophistication (see p. 18 of the Operations and Algebraic Thinking Progression for similar two-step problems involving discrete objects). Conversely, "missing measurements" problems about length may be presented with diagrams.

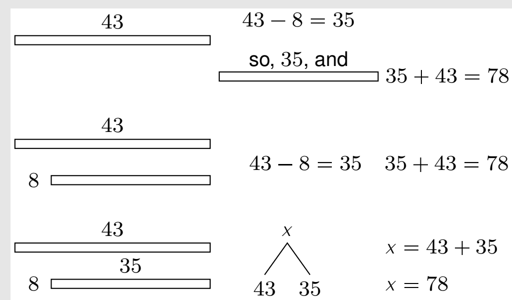
These understandings are essential in supporting work with number line diagrams.[2.MD.6] That is, to use a number line diagram to understand number and number operations, students need to understand that number line diagrams have specific conventions: the use of a single position to represent a whole number and the use of marks to indicate those positions. They need to understand that a number line diagram is like a ruler in that consecutive whole numbers are 1 unit apart, thus they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie students' successful use of number line diagrams. Students think of a number line diagram as a measurement model and use strategies relating to distance, proximity of numbers, and reference points.

After experience with measuring, second graders learn to estimate lengths.[2.MD.3] Real-world applications of length often involve estimation. Skilled estimators move fluently back and forth between written or verbal length measurements and representations of their corresponding magnitudes on a *mental ruler* (also called the "mental number line"). Although having real-world "benchmarks" is useful

[2.MD.4] Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

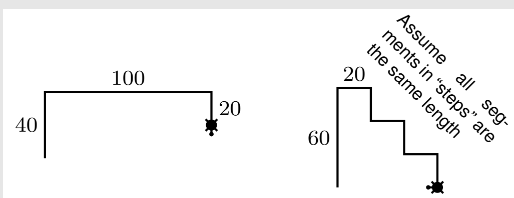
[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

Missing measurements problems



Different solution methods for "A girl had a 43 cm section of a necklace and another section that was 8 cm shorter than the first. How long the necklace would be if she combined the two sections?" [2.MD.5]

Missing measurements problems



What are the missing lengths of the third and fourth sides of the rectangle?

What are the missing lengths of each step and the bottom of the stairway?

These problems might be presented in the context of turtle geometry. Students work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house (the remainder of the right side, and the bottom). They then type in Logo commands (e.g., for the rectangle, forward 40 right 90 fd 100 rt 90 fd 20 fd 20 rt 90 fd 100) to check their calculations (MP5).

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.3] Estimate lengths using units of inches, feet, centimeters, and meters.

(e.g., a meter is about the distance from the floor to the top of a doorknob), instruction should also help children build understandings of scales and concepts of measurement into their estimation competencies. Although "guess and check" experiences can be useful, research suggests explicit teaching of estimation strategies (such as iteration of a mental image of the unit or comparison with a known measurement) and prompting students to learn reference or benchmark lengths (e.g., an inch-long piece of gum, a 6-inch dollar bill), order points along a continuum, and build up mental rulers.

Length measurement should also be used in other domains of mathematics, as well as in other subjects, such as science, and connections should be made where possible. For example, a line plot scale is just a ruler, usually with a non-standard unit of length. Teachers can ask students to discuss relationships they see between rulers and line plot scales. Data using length measures might be graphed (see example on pp. 8–9 of the Measurement Data Progression). Students could also graph the results of many students measuring the same object as precisely as possible (even involving halves or fourths of a unit) and discuss what the "real" measurement of the object might be. Emphasis on students solving real measurement problems, and, in so doing, building and iterating units, as well as units of units, helps students development strong concepts and skills. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics.

Area and volume: Foundations To learn area (and, later, volume) concepts and skills meaningfully in later grades, students need to develop the ability known as *spatial structuring*. Students need to be able to see a rectangular region as decomposable into rows and columns of squares. This competence is discussed in detail in the Geometry Progression, but is mentioned here for two reasons. First, such spatial structuring precedes meaningful mathematical use of the structures, such as determining area or volume. Second, Grade 2 work in multiplication involves work with rectangular arrays,[2.G.2] and this work is an ideal context in which to simultaneously develop both arithmetical and spatial structuring foundations for later work with area.

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Grade 3

Perimeter Third graders focus on solving real-world and mathematical problems involving perimeters of polygons.[3.MD.8] A perimeter is the boundary of a two-dimensional shape. For a polygon, the length of the perimeter is the sum of the lengths of the sides. Initially, it is useful to have sides marked with unit length marks, allowing students to count the unit lengths. Later, the lengths of the sides can be labeled with numerals. As with all length tasks, students need to count the length-units and not the end-points. Next, students learn to mark off unit lengths with a ruler and label the length of each side of the polygon. For rectangles, parallelograms, and regular polygons, students can discuss and justify faster ways to find the perimeter length than just adding all of the lengths (MP3). Rectangles and parallelograms have opposite sides of equal length, so students can double the lengths of adjacent sides and add those numbers or add lengths of two adjacent sides and double that number. A regular polygon has all sides of equal length, so its perimeter length is the product of one side length and the number of sides.

Perimeter problems for rectangles and parallelograms often give only the lengths of two adjacent sides or only show numbers for these sides in a drawing of the shape. The common error is to add just those two numbers. Having students first label the lengths of the other two sides as a reminder is helpful.

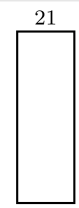
Students then find unknown side lengths in more difficult "missing measurements" problems and other types of perimeter problems.[3.MD.8]

Children learn to subdivide length-units. Making one's own ruler and marking halves and other partitions of the unit may be helpful in this regard. For example, children could fold a unit in halves, mark the fold as a half, and then continue to do so, to build fourths and eighths, discussing issues that arise. Such activities relate to fractions on the number line.[3.NF.2] Labeling all of the fractions can help students understand rulers marked with halves and fourths but not labeled with these fractions. Students also measure lengths using rulers marked with halves and fourths of an inch.[3.MD.4] They show these data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters (see the Measurement Data Progression, p. 10).

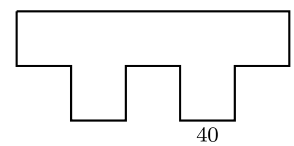
Understand concepts of area and relate area to multiplication and to addition Third graders focus on learning area. Students learn formulas to compute area, with those formulas based on, and summarizing, a firm conceptual foundation about what area is. Students need to learn to conceptualize area as the amount of two-dimensional space in a bounded region and to measure it by choosing a unit of area, often a square. A two-dimensional geometric figure that is covered by a certain number of squares without gaps

[3.MD.8] Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Missing measurements and other perimeter problems



The perimeter of this rectangle is 168 length units. What are the lengths of the three unlabeled sides?



Assume all short segments are the same length and all angles are right

Compare these problems with the "missing measurements" problems of Grade 2.

Another type of perimeter problem is to draw a robot on squared grid paper that meets specific criteria. All the robot's body parts must be rectangles. The perimeter of the head might be 36 length-units, the body, 72; each arm, 24; and each leg, 72. Students are asked to provide a convincing argument that their robots meet these criteria (MP3). Next, students are asked to figure out the area of each of their body parts (in square units). These are discussed, with students led to reflect on the different areas that may be produced with rectangles of the same perimeter. These types of problems can be also presented as turtle geometry problems. Students create the commands on paper and then give their commands to the Logo turtle to check their calculations. For turtle length units, the perimeter of the head might be 300 length-units, the body, 600; each arm, 400; and each leg, 640.

[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

[3.MD.4] Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

or overlaps can be said to have an area of that number of square units.[3.MD.5]

Activities such as those in the Geometry Progression teach students to compose and decompose geometric regions. To begin an explicit focus on area, teachers might then ask students which of three rectangles covers the most area. Students may first solve the problem with decomposition (cutting and/or folding) and re-composition, and eventually analyses with area-units, by covering each with unit squares (tiles).[3.MD.5, 3.MD.6] Discussions should clearly distinguish the attribute of area from other attributes, notably length.

Students might then find the areas of other rectangles. As previously stated, students can be taught to multiply length measurements to find the area of a rectangular region. But, in order that they make sense of these quantities (MP2), they first learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows.[3.MD.7a] This relies on the development of spatial structuring.[MP7] To build from spatial structuring to understanding the number of area-units as the product of number of units in a row and number of rows, students might draw rectangular arrays of squares and learn to determine the number of squares in each row with increasingly sophisticated strategies, such as skip-counting the number in each row and eventually multiplying the number in each row by the number of rows (MP8). They learn to partition a rectangle into identical squares by anticipating the final structure and forming the array by drawing line segments to form rows and columns. They use skip counting and multiplication to determine the number of squares in the array.

Many activities that involve seeing and making arrays of squares to form a rectangle might be needed to build robust conceptions of a rectangular area structured into squares. One such activity is illustrated in the margin. In this progression, less sophisticated activities of this sort were suggested for earlier grades so that Grade 3 students begin with some experience.

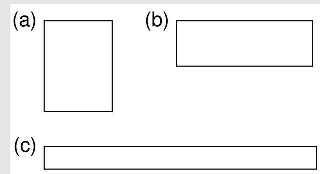
Students learn to understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior (MP3).[3.MD.7a] For example, students might explain that one length tells how many unit squares in a row and the other length tells how many rows there are.

Students might then solve numerous problems that involve rectangles of different dimensions (e.g., designing a house with rooms that fit specific area criteria) to practice using multiplication to compute areas.[3.MD.7b] The areas involved should not all be rectangular, but decomposable into rectangles (e.g., an "L-shaped" room).[3.MD.7d]

Students also might solve problems such as finding all the rectangular regions with whole-number side lengths that have an area of 12 area-units, doing this later for larger rectangles (e.g., enclosing 24, 48, or 72 area-units), making sketches rather than drawing each

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

Which rectangle covers the most area?



These rectangles are formed from unit squares (tiles students have used) although students are not informed of this or the rectangle's dimensions: (a) 4 by 3, (b) 2 by 6, and (c) 1 row of 12. Activity from Lehrer, et al., 1998, "Developing understanding of geometry and space in the primary grades," in R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*, Lawrence Erlbaum Associates.

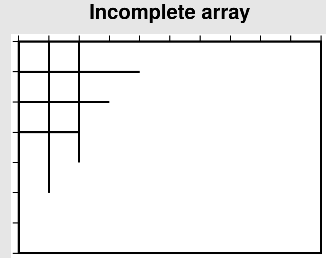
3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

MP7 See the Geometry Progression

Incomplete array



To determine the area of this rectangular region, students might be encouraged to construct a row, corresponding to the indicated positions, then repeating that row to fill the region. Cutouts of strips of rows can help the needed spatial structuring and reduce the time needed to show a rectangle as rows or columns of squares. Drawing all of the squares can also be helpful, but it is slow for larger rectangles. Drawing the unit lengths on the opposite sides can help students see that joining opposite unit end-points will create the needed unit square grid.

3.MD.7b Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

3.MD.7d Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

square. They learn to justify their belief they have found all possible solutions (MP3).

Similarly using concrete objects or drawings, and their competence with composition and decomposition of shapes, spatial structuring, and addition of area measurements, students learn to investigate arithmetic properties using area models. For example, they learn to rotate rectangular arrays physically and mentally, understanding that their areas are preserved under rotation, and thus, for example, $4 \times 7 = 7 \times 4$, illustrating the commutative property of multiplication.[3.MD.7c] They also learn to understand and explain that the area of a rectangular region of, for example, 12 length-units by 5 length-units can be found either by multiplying 12×5, or by adding two products, e.g., 10×5 and 2×5, illustrating the distributive property.

3.MD.7c Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures With strong and distinct concepts of both perimeter and area established, students can work on problems to differentiate their measures. For example, they can find and sketch rectangles with the same perimeter and different areas or with the same area and different perimeters and justify their claims (MP3).[3.MD.8] Differentiating perimeter from area is facilitated by having students draw congruent rectangles and measure, mark off, and label the unit lengths all around the perimeter on one rectangle, then do the same on the other rectangle but also draw the square units. This enables students to see the units involved in length and area and find patterns in finding the lengths and areas of non-square and square rectangles (MP7). Students can continue to describe and show the units involved in perimeter and area after they no longer need these .

3.MD.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Problem solving involving measurement and estimation of intervals of time, liquid volumes, and masses of objects Students in Grade 3 learn to solve a variety of problems involving measurement and such attributes as length and area, liquid volume, mass, and time.[3.MD.1, 3.MD.2] Many such problems support the Grade 3 emphasis on multiplication (see Table 1) and the mathematical practices of making sense of problems (MP1) and representing them with equations, drawings, or diagrams (MP4). Such work will involve units of mass such as the kilogram.

3.MD.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

3.MD.2 Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[2] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[3]

Table 1: Multiplication and division situations for measurement

	Unknown Product $A \times B = \square$	Group Size Unknown $A \times \square = C$ and $C \div A = \square$	Number of Groups Unknown $\square \times B = C$ and $C \div B = \square$
Grouped Objects (Units of Units)	You need A lengths of string, each B inches long. How much string will you need altogether?	You have C inches of string, which you will cut into A equal pieces. How long will each piece of string be?	You have C inches of string, which you will cut into pieces that are B inches long. How many pieces of string will you have?
Arrays of Objects (Spatial Structuring)	What is the area of a A cm by B cm rectangle?	A rectangle has area C square centimeters. If one side is A cm long, how long is a side next to it?	A rectangle has area C square centimeters. If one side is B cm long, how long is a side next to it?
Compare	A rubber band is B cm long. How long will the rubber band be when it is stretched to be A times as long?	A rubber band is stretched to be C cm long and that is A times as long as it was at first. How long was the rubber band at first?	A rubber band was B cm long at first. Now it is stretched to be C cm long. How many times as long is the rubber band now as it was at first?

Adapted from box 2-4 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33. Note that Grade 3 work does not include Compare problems with "times as much," see the Operations and Algebraic Thinking Progression, Table 3, also p. 29.

A few words on volume are relevant. Compared to the work in area, volume introduces more complexity, not only in adding a third dimension and thus presenting a significant challenge to students' spatial structuring, but also in the materials whose volumes are measured. These materials may be solid or fluid, so their volumes are generally measured with one of two methods, e.g., "packing" a right rectangular prism with cubic units or "filling" a shape such as a right circular cylinder. Liquid measurement, for many third graders, may be limited to a one-dimensional unit structure (i.e., simple iterative counting of height that is not processed as three-dimensional). Thus, third graders can learn to measure with liquid volume and to solve problems requiring the use of the four arithmetic operations, when liquid volumes are given in the same units throughout each problem. Because liquid measurement can be represented with one-dimensional scales, problems may be presented with drawings or diagrams, such as measurements on a beaker with a measurement scale in milliliters.

Grade 4

In Grade 4, students build on competencies in measurement and in building and relating units and units of units that they have developed in number, geometry, and geometric measurement.

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit Fourth graders learn the relative sizes of measurement units within a system of measurement[4.MD.1] including:

> *length*: meter (m), kilometer (km), centimeter (cm), millimeter (mm); volume: liter (l), milliliter (ml, 1 cubic centimeter of water; a liter, then, is 1000 ml);
>
> *mass*: gram (g, about the weight of a cc of water), kilogram (kg); time: hour (hr), minute (min), second (sec).

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

Super- or subordinate unit	Length in terms of basic unit
kilometer	10^3 or 1000 meters
hectometer	10^2 or 100 meters
decameter	10^1 or 10 meters
meter	1 meter
decimeter	10^{-1} or $\frac{1}{10}$ meters
centimeter	10^{-2} or $\frac{1}{100}$ meters
millimeter	10^{-3} or $\frac{1}{1000}$ meters

Note the similarity to the structure of base-ten units and U.S. currency (see illustrations on p. 12 of the Number and Operations in Base Ten Progression).

For example, students develop benchmarks and mental images about a meter (e.g., about the height of a tall chair) and a kilometer (e.g., the length of 10 football fields including the end zones, or the distance a person might walk in about 12 minutes), and they also understand that "kilo" means a thousand, so 3000 m is equivalent to 3 km.

Expressing larger measurements in smaller units within the metric system is an opportunity to reinforce notions of place value. There are prefixes for multiples of the basic unit (meter or gram), although only a few (kilo-, centi-, and milli-) are in common use. Tables such as the one in the margin indicate the meanings of the prefixes by showing them in terms of the basic unit (in this case, meters). Such tables are an opportunity to develop or reinforce place value concepts and skills in measurement activities.

Relating units within the metric system is another opportunity to think about place value. For example, students might make a table that shows measurements of the same lengths in centimeters and meters.

Centimeter and meter equivalences

cm	m
100	1
200	2
300	3
500	
1000	

Foot and inch equivalences

feet	inches
0	0
1	12
2	24
3	

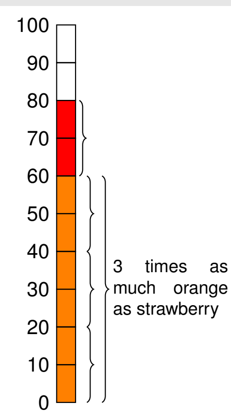
Relating units within the traditional system provides an opportunity to engage in mathematical practices, especially "look for and make use of structure" (MP7) and "look for and express regularity in repeated reasoning" (MP8). For example, students might make a table that shows measurements of the same lengths in feet and inches.

Students also combine competencies from different domains as they solve measurement problems using all four arithmetic operations, addition, subtraction, multiplication, and division (see examples in Table 1).[4.MD.2] For example, "How many liters of juice does the class need to have at least 35 cups if each cup takes 225 ml?" Students may use tape or number line diagrams for solving such problems (MP1).

4.MD.2 Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Using tape diagrams to solve word problems

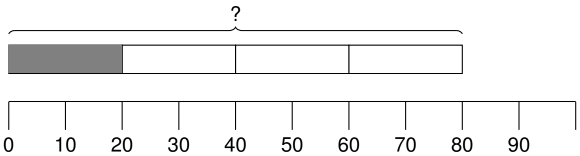
Lisa put two flavors of soda in a glass. There were 80 ml of soda in all. She put three times as much orange drink as strawberry. How many ml of orange did she put in?



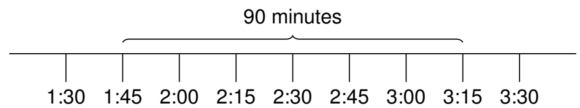
In this diagram, quantities are represented on a measurement scale.

Using number line diagrams to solve word problems

Juan spent 1/4 of his money on a game. The game cost \$20. How much money did he have at first?



What time does Marla have to leave to be at her friend's house by a quarter after 3 if the trip takes 90 minutes?



Using a number line diagram to represent time is easier if students think of digital clocks rather than round clocks. In the latter case, placing the numbers on the number line involves considering movements of the hour and minute hands.

Students learn to consider perimeter and area of rectangles, begun in Grade 3, more abstractly (MP2). Based on work in previous grades with multiplication, spatially structuring arrays, and area, they abstract the formula for the area of a rectangle $A = l \times w$.•

Students generate and discuss advantages and disadvantages of various formulas for the perimeter length of a rectangle that is l units by w units.• Giving verbal summaries of these formulas is also helpful. For example, a verbal summary of the basic formula, $A = l + w + l + w$, is "add the lengths of all four sides." Specific numerical instances of other formulas or mental calculations for the perimeter of a rectangle can be seen as examples of the properties of operations, e.g., $2l + 2w = 2(l + w)$ illustrates the distributive property.

Perimeter problems often give only one length and one width, thus remembering the basic formula can help to prevent the usual error of only adding one length and one width. The formula $P = 2(l + w)$ emphasizes the step of multiplying the total of the given lengths by 2. Students can make a transition from showing all length units along the sides of a rectangle or all area units within (as in Grade 3, p. 18) by drawing a rectangle showing just parts of these as a reminder of which kind of unit is being used. Writing all of the lengths around a rectangle can also be useful. Discussions of formulas such as $P = 2l + 2w$, can note that unlike area formulas, perimeter formulas combine length measurements to yield a length measurement.

Such abstraction and use of formulas underscores the importance of distinguishing between area and perimeter in Grade 3[3.MD.8] and maintaining the distinction in Grade 4 and later grades, where rectangle perimeter and area problems may get more complex and problem solving can benefit from knowing or being able to rapidly remind oneself of how to find an area or perimeter. By repeatedly reasoning about how to calculate areas and perimeters of rectangles, students can come to see area and perimeter formulas as summaries of all such calculations (MP8).

• The formula is a generalization of the understanding, that, given a unit of length, a rectangle whose sides have length w units and l units, can be partitioned into w rows of unit squares with l squares in each row. The product $l \times w$ gives the number of unit squares in the partition, thus the area measurement is $l \times w$ square units. These square units are derived from the length unit.

• For example, $P = 2l + 2w$ has two multiplications and one addition, but $P = 2(l + w)$, which has one addition and one multiplication, involves fewer calculations. The latter formula is also useful when generating all possible rectangles with a given perimeter. The length and width vary across all possible pairs whose sum is half of the perimeter (e.g., for a perimeter of 20, the length and width are all of the pairs of numbers with sum 10).

3.MD.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Students learn to apply these understandings and formulas to the solution of real-world and mathematical problems.[4.MD.3] For example, they might be asked, "A rectangular garden has as an area of 80 square feet. It is 5 feet wide. How long is the garden?" Here, specifying the area and the width, creates an unknown factor problem (see Table 1). Similarly, students could solve perimeter problems that give the perimeter and the length of one side and ask the length of the adjacent side. Students could be challenged to solve multistep problems such as the following. "A plan for a house includes rectangular room with an area of 60 square meters and a perimeter of 32 meters. What are the length and the width of the room?"

4.MD.3 Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

In Grade 4 and beyond, the mental visual images for perimeter and area from Grade 3 can support students in problem solving with these concepts. When engaging in the mathematical practice of reasoning abstractly and quantitatively (MP2) in work with area and perimeter, students think of the situation and perhaps make a drawing. Then they recreate the "formula" with specific numbers and one unknown number as a situation equation for this particular numerical situation.• "Apply the formula" does not mean write down a memorized formula and put in known values because at Grade 4 students do not evaluate expressions (they begin this type of work in Grade 6). In Grade 4, working with perimeter and area of rectangles is still grounded in specific visualizations and numbers. These numbers can now be any of the numbers used in Grade 4 (for addition and subtraction for perimeter and for multiplication and division for area).[4.NBT.4, 4.NF.3d, 4.OA.4] By repeatedly reasoning about constructing situation equations for perimeter and area involving specific numbers and an unknown number, students will build a foundation for applying area, perimeter, and other formulas by substituting specific values for the variables in later grades.

• "Situation equation" refers to the idea that the student constructs an equation as a representation of a situation rather than identifying the situation as an example of a familiar equation.

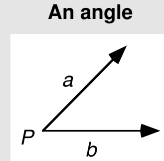
4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

4.NF.3d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

4.OA.4 Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Understand concepts of angle and measure angles Angle measure is a "turning point" in the study of geometry. Students often find angles and angle measure to be difficult concepts to learn, but that learning allows them to engage in interesting and important mathematics. An *angle* is the union of two rays, a and b, with the same initial point P. The rays can be made to coincide by rotating one to the other about P; this rotation determines the size of the angle between a and b. The rays are sometimes called the *sides* of the angles.

An angle



P is called the *vertex* of the angle and the rays a and b are called the *arms*.

Another way of saying this is that each ray determines a direction and the angle size measures the change from one direction to the other. (This illustrates how angle measure is related to the concepts of parallel and perpendicular lines in Grade 4 geometry.) A clockwise rotation is considered positive in surveying or turtle geometry; but a counterclockwise rotation is considered positive in Euclidean geometry. Angles are measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and degrees are the unit used to measure angles in elementary school. A full rotation is thus 360°.

Two angles are called *complementary* if their measurements have the sum of 90°. Two angles are called *supplementary* if their measurements have the sum of 180°. Two angles with the same vertex that overlap only at a boundary (i.e., share a side) are called *adjacent angles*.

Like length, area, and volume, angle measure is additive: The sum of the measurements of *adjacent angles* is the measurement of the angle formed by their union. This leads to other important properties. If a right angle is decomposed into two adjacent angles, the sum is 90°, thus they are complementary. Two adjacent angles that compose a "straight angle" of 180° must be supplementary. In some situations (see margin), such properties allow logical progressions of statements (MP3).

As with all measureable attributes, students must first recognize the attribute of angle measure, and distinguish it from other attributes. This may not appear too difficult, as the measure of angles and rotations appears to knowledgeable adults as quite different than attributes such as length and area. However, the unique nature of angle size leads many students to initially confuse angle measure with other, more familiar, attributes. Even in contexts designed to evoke a dynamic image of turning, such as hinges or doors, many students use the length between the endpoints, thus teachers find it useful to repeatedly discuss such cognitive "traps."

As with other concepts (e.g., see the Geometry Progression), students need varied examples and explicit discussions to avoid learning limited ideas about measuring angles (e.g., misconceptions that a right angle is an angle that points to the right, or two right angles represented with different orientations are not equal in measure). If examples and tasks are not varied, students can develop incomplete and inaccurate notions. For example, some come to associate all slanted lines with 45° measures and horizontal and vertical lines with measures of 90°. Others believe angles can be "read off" a protractor in "standard" position, that is, a base is horizontal, even if neither arm of the angle is horizontal. Measuring and then sketching many angles with no horizontal or vertical arms,[4.MD.6] perhaps initially using circular 360° protractors, can help students avoid such limited conceptions.

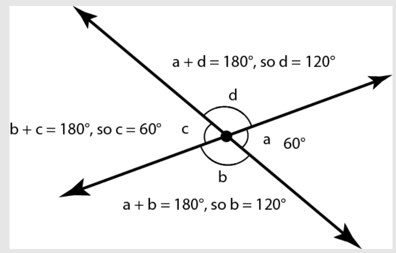
As with length, area, and volume, children need to understand equal partitioning and unit iteration to understand angle and turn measure. Whether defined as more statically as the measure of the figure formed by the intersection of two rays or as turning, having a given angle measure involves a relationship between components of plane figures and therefore is a *property* (see the Overview in the Geometry Progression).[4.G.2]

Given the complexity of angles and angle measure, it is unsur-

An angle

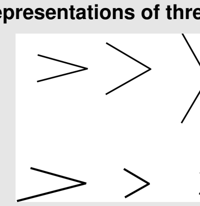
name	measurement
right angle	90°
straight angle	180°
acute angle	between 0 and 90°
obtuse angle	between 90° and 180°
reflex angle	between 180° and 360°

Angles created by the intersection of two lines



When two lines intersect, they form four angles. If the measurement of one is known (e.g., angle a is 60°), the measurement of the other three can be determined.

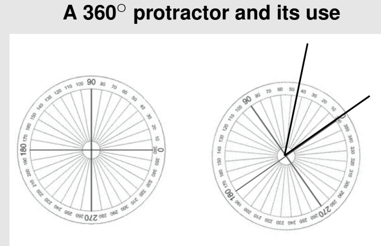
Two representations of three angles



Initially, some students may correctly compare angle sizes only if all the line segments are the same length (as shown in the top row). If the lengths of the line segments are different (as shown in the bottom row), these students base their judgments on the lengths of the segments, the distances between their endpoints, or even the area of the triangles determined by the drawn arms. They believe that the angles in the bottom row decrease in size from left to right, although they have, respectively, the same angle measurements as those in the top row.

4.MD.6 Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.

A 360° protractor and its use



The figure on the right shows a protractor being used to measure a 45° angle. The protractor is placed so that one side of the angle lies on the line corresponding to 0° on the protractor and the other side of the angle is located by a clockwise rotation from that line.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

prising that students in the early and elementary grades often form separate concepts of angles as figures and turns, and may have separate notions for different turn contexts (e.g., unlimited rotation as a fan vs. a hinge) and for various "bends."

However, students can develop more accurate and useful angle and angle measure concepts if presented with angles in a variety of situations. They learn to find the common features of superficially different situations such as turns in navigation, slopes, bends, corners, and openings. With guidance, they learn to represent an angle in any of these contexts as two rays, even when both rays are not explicitly represented in the context; for example, the horizontal or vertical in situations that involve slope (e.g., roads or ramps), or the angle determined by looking up from the horizon to a tree- or mountain-top. Eventually they abstract the common attributes of the situations as angles (which are represented with rays and a vertex, MP4) and angle measurements (MP2). To accomplish the latter, students integrate turns, and a general, dynamic understanding of angle measure-as-rotation, into their understandings of angles-as-objects. Computer manipulatives and tools can help children bring such a dynamic concept of angle measure to an explicit level of awareness. For example, dynamic geometry environments can provide multiple linked representations, such as a screen drawing that students can "drag" which is connected to a numerical representation of angle size. Games based on similar notions are particularly effective when students manipulate not the arms of the angle itself, but a representation of rotation (a small circular diagram with radii that, when manipulated, change the size of the target angle turned).

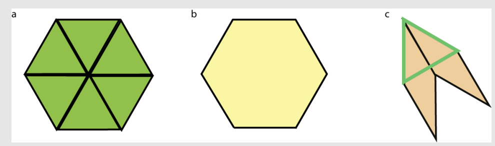
Students with an accurate conception of angle can recognize that angle measure is *additive*.[4.MD.7] As with length, area, and volume, when an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Students can then solve interesting and challenging addition and subtraction problems to find the measurements of unknown angles on a diagram in real world and mathematical problems. For example, they can find the measurements of angles formed a pair of intersecting lines, as illustrated above, or given a diagram showing the measurement of one angle, find the measurement of its complement. They can use a protractor to check, not to check their reasoning, but to ensure that they develop full understanding of the mathematics and mental images for important benchmark angles (e.g., 30°, 45°, 60°, and 90°).

Such reasoning can be challenged with many situations as illustrated in the margin.

Similar activities can be done with drawings of shapes using right angles and half of a right angle to develop the important benchmarks of 90° and 45°.

Missing measures can also be done in the turtle geometry context, building on the previous work. Note that unguided use of Logo's turtle geometry does not necessary develop strong angle

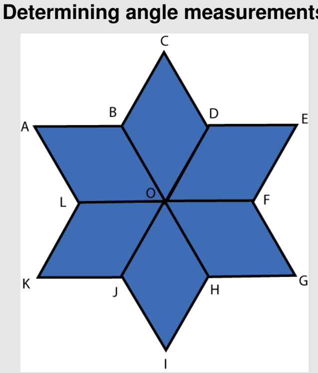
Determining angles in pattern blocks



Students might determine all the angles in the common "pattern block" shape set based on equilateral triangles. Placing six equilateral triangles so that they share a common vertex (as shown in part a), students can figure out that because the sum of the angles at this vertex is 360°, each angle which shares this vertex must have measure 60°. Because they are congruent, all the angles of the equilateral triangles must have measure 60° (again, to ensure they develop a firm foundation, students can verify these for themselves with a protractor). Because each angle of the regular hexagon (part b) is composed of two angles from equilateral triangles, the hexagon's angles each measure 120°. Similarly, in a pattern block set, two of the smaller angles from tan rhombi compose an equilateral triangle's angle, so each of the smaller rhombus angles has measure 30°.

[4.MD.7] Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Determining angle measurements

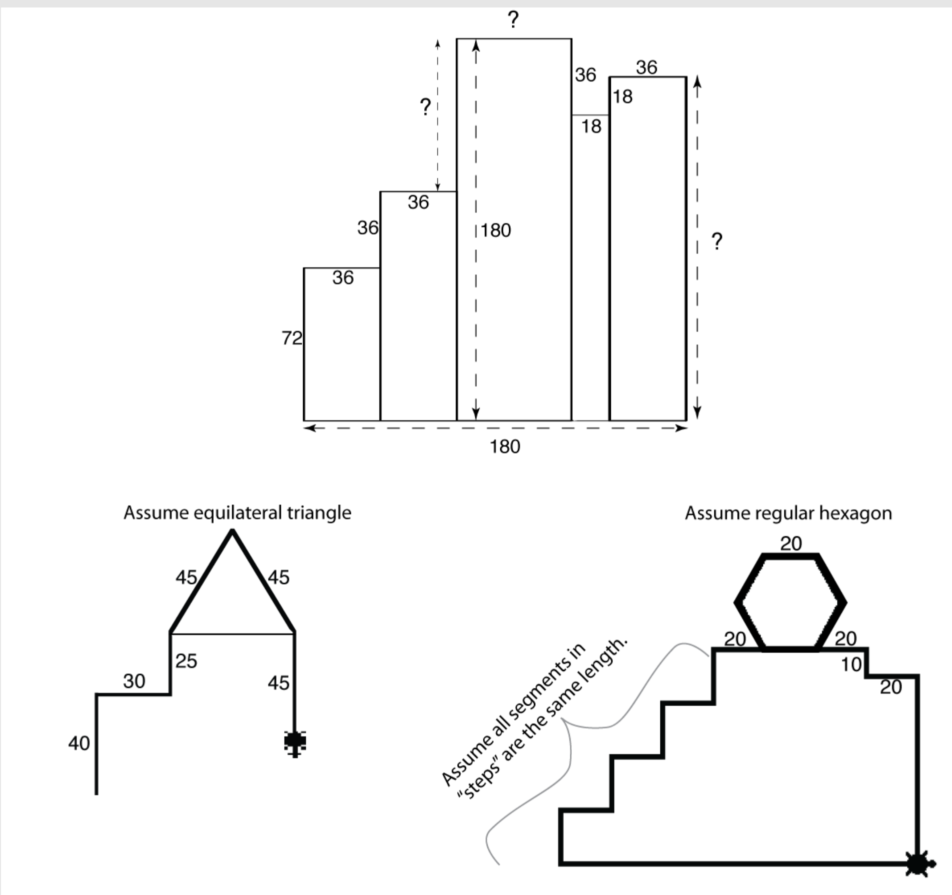


Students might be asked to determine the measurements of the following angles:
$\angle BOD$
$\angle BOF$
$\angle ODE$
$\angle CDE$
$\angle CDJ$
$\angle BHG$

concepts. However, if teachers emphasize mathematical tasks and, within those tasks, the difference between the angle of rotation the turtle makes (in a polygon, the external angle) and the angle formed (internal angle) and integrates the two, students can develop accurate and comprehensive understandings of angle measure. For example, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such questions help to connect what are often initially isolated ideas about angle conceptions.

These understandings support students in finding all the missing length and angle measures in situations such as the examples in the margin (compare to the missing measures problems Grade 2 and Grade 3).

Missing measures: Length (top) and length and angle (turn)



Students are asked to determine the missing lengths. They might first work on paper to figure out how far the Logo turtle would have to travel to *finish* drawing the house, then type in Logo commands to verify their reasoning and calculations.

Grade 5

Convert like measurement units within a given measurement system In Grade 5, students extend their abilities from Grade 4 to express measurements in larger or smaller units within a measurement system.[4.MD.1, 5.MD.1] This is an excellent opportunity to reinforce notions of place value for whole numbers and decimals, and connection between fractions and decimals (e.g., $2\,\frac{1}{2}$ meters can be expressed as 2.5 meters or 250 centimeters). For example, building on the table from Grade 4, Grade 5 students might complete a table of equivalent measurements in feet and inches.

Grade 5 students also learn and use such conversions in solving multi-step, real world problems (see example in the margin).

Understand concepts of volume and relate volume to multiplication and to addition The major emphasis for measurement in Grade 5 is volume. Volume not only introduces a third dimension and thus a significant challenge to students' spatial structuring, but also complexity in the nature of the materials measured. That is, solid units are "packed," such as cubes in a three-dimensional array, whereas a liquid "fills" three-dimensional space, taking the shape of the container. As noted earlier (see Overview, also Grades 1 and 3), the unit structure for liquid measurement may be psychologically one-dimensional for some students.

"Packing" volume is more difficult than iterating a unit to measure length and measuring area by tiling. Students learn about a unit of volume, such as a cube with a side length of 1 unit, called a unit cube.[5.MD.3] They pack cubes (without gaps) into right rectangular prisms and count the cubes to determine the volume or build right rectangular prisms from cubes and see the layers as they build.[5.MD.4] They can use the results to compare the volume of right rectangular prisms that have different dimensions. Such experiences enable students to extend their spatial structuring from two to three dimensions (see the Geometry Progression). That is, they learn to both mentally decompose and recompose a right rectangular prism built from cubes into layers, each of which is composed of rows and columns. That is, given the prism, they have to be able to decompose it, understanding that it can be partitioned into layers, and each layer partitioned into rows, and each row into cubes. They also have to be able to compose such as structure, multiplicatively, back into higher units. That is, they eventually learn to conceptualize a layer as a unit that itself is composed of units of units—rows, each row composed of individual cubes—and they iterate that structure. Thus, they might predict the number of cubes that will be needed to fill a box given the net of the box.

Another complexity of volume is the connection between "packing" and "filling." Often, for example, students will respond that a box can be filled with 24 centimeter cubes, or build a structure of 24 cubes, and still think of the 24 as individual, often discrete, not

4.MD.1 Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

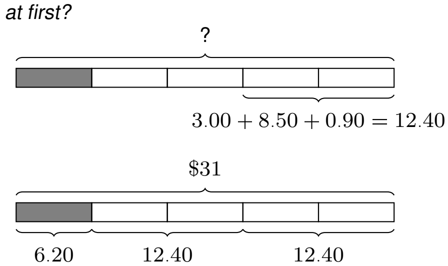
5.MD.1 Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Feet	Inches
0	0
	1
	2
	3

In Grade 6, this table can be discussed in terms of ratios and proportional relationships (see the Ratio and Proportion Progression). In Grade 5, however, the main focus is on arriving at the measurements that generate the table.

Multi-step problem with unit conversion

Kumi spent a fifth of her money on lunch. She then spent half of what remained. She bought a card game for $3, a book for $8.50, and candy for 90 cents. How much money did she have at first?

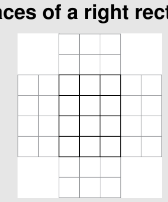


Students can use tape diagrams to represent problems that involve conversion of units, drawing diagrams of important features and relationships (MP1).

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

Net for five faces of a right rectangular prism



Students are given a net and asked to predict the number of cubes required to fill the container formed by the net. In such tasks, students may initially count single cubes or repeatedly add the number of cubes in a row to determine the number in each layer, and repeatedly add the number in each layer to find the total number of unit cubes. In folding the net to make the shape, students can see how the side rectangles fit together and determine the number of layers.

necessarily *units of volume*. They may, for example, not respond confidently and correctly when asked to fill a graduated cylinder marked in cubic centimeters with the amount of liquid that would fill the box. That is, they have not yet connected their ideas about filling volume with those concerning packing volume. Students learn to move between these conceptions, e.g., using the same container, both filling (from a graduated cylinder marked in ml or cc) and packing (with cubes that are each 1 cm^3). Comparing and discussing the volume-units and what they represent can help students learn a general, complete, and interconnected conceptualization of volume as filling three-dimensional space.

Students then learn to determine the volumes of several right rectangular prisms, using cubic centimeters, cubic inches, and cubic feet. With guidance, they learn to increasingly apply multiplicative reasoning to determine volumes, looking for and making use of structure (MP7). That is, they understand that multiplying the length times the width of a right rectangular prism can be viewed as determining how many cubes would be in each layer if the prism were packed with or built up from unit cubes.[5.MD.5a] They also learn that the height of the prism tells how many layers would fit in the prism. That is, they understand that volume is a derived attribute that, once a length unit is specified, can be computed as the product of three length measurements or as the product of one area and one length measurement.

Then, students can learn the formulas $V = l \times w \times h$ and $V = B \times h$ for right rectangular prisms as efficient methods for computing volume, maintaining the connection between these methods and their previous work with computing the number of unit cubes that pack a right rectangular prism.[5.MD.5b] They use these competencies to find the volumes of right rectangular prisms with edges whose lengths are whole numbers and solve real-world and mathematical problems involving such prisms.

Students also recognize that volume is additive (see Overview) and they find the total volume of solid figures composed of two right rectangular prisms.[5.MD.5c] For example, students might design a science station for the ocean floor that is composed of several rooms that are right rectangular prisms and that meet a set criterion specifying the total volume of the station. They draw their station (e.g., using an isometric grid, MP7) and justify how their design meets the criterion (MP1).

5.MD.5a Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

5.MD.5b Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

5.MD.5c Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Where the Geometric Measurement Progression is heading

Connection to Geometry In Grade 6, students build on their understanding of length, area, and volume measurement, learning to how to compute areas of right triangles and other special figures and volumes of right rectangular prisms that do not have measurements given in whole numbers. To do this, they use dissection arguments. These rely on the understanding that area and volume measures are additive, together with decomposition of plane and solid shapes (see the K–5 Geometry Progression) into shapes whose measurements students already know how to compute (MP1, MP7). In Grade 7, they use their understanding of length and area in learning and using formulas for the circumference and area of circles. In Grade 8, they use their understanding of volume in learning and using formulas for the volumes of cones, cylinders, and spheres. In high school, students learn formulas for volumes of pyramids and revisit the formulas from Grades 7 and 8, explaining them with dissection arguments, Cavalieri's principle, and informal limit arguments.

Connection to the Number System In Grade 6, understanding of length-units and spatial structuring comes into play as students learn to plot points in the coordinate plane.

Connection to Ratio and Proportion Students use their knowledge of measurement and units of measurement in Grades 6–8, coming to see conversions between two units of measurement as describing proportional relationships.

Progressions for the Common Core State Standards in Mathematics (draft)

19 September 2013

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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

Chapter 1

Geometry, K–6

Overview

Like core knowledge of number, core geometrical knowledge appears to be a universal capability of the human mind. Geometric and spatial thinking are important in and of themselves, because they connect mathematics with the physical world, and play an important role in modeling phenomena whose origins are not necessarily physical, for example, as networks or graphs. They are also important because they support the development of number and arithmetic concepts and skills. Thus, geometry is essential for all grade levels for many reasons: its mathematical content, its roles in physical sciences, engineering, and many other subjects, and its strong aesthetic connections.

This progression discusses the most important goals for elementary geometry according to three categories.

- Geometric shapes, their components (e.g., sides, angles, faces), their properties, and their categorization based on those properties.
- Composing and decomposing geometric shapes.
- Spatial relations and spatial structuring.

Geometric shapes, components, and properties. Students develop through a series of levels of geometric and spatial thinking. As with all of the domains discussed in the Progressions, this development depends on instructional experiences. Initially, students cannot reliably distinguish between examples and nonexamples of categories of shapes, such as triangles, rectangles, and squares.• With experience, they progress to the next level of thinking, recognizing shapes in ways that are visual or syncretic (a fusion of differing systems). At this level, students can recognize shapes as wholes, but cannot form mathematically-constrained mental images of them. A given figure is a rectangle, for example, because "it looks like a door." They do not explicitly think about the components or

• In formal mathematics, a geometric shape is a boundary of a region, e.g., "circle" is the boundary of a disk. This distinction is not expected in elementary school.

about the defining attributes, or properties, of shapes. Students then move to a descriptive level in which they can think about the components of shapes, such as triangles having three sides. For example, kindergartners can decide whether all of the sides of a shape are straight and they can count the sides. They also can discuss if the shape is closed• and thus convince themselves that a three-sided shape is a triangle even if it is "very skinny" (e.g., an isosceles triangle with large obtuse angle).

At the analytic level, students recognize and characterize shapes by their *properties*.[1] For instance, a student might think of a square as a figure that has four equal sides and four right angles. Different components of shapes are the focus at different grades, for instance, second graders measure lengths and fourth graders measure angles (see the Geometric Measurement Progression). Students find that some combinations of properties signal certain classes of figures and some do not; thus the seeds of geometric implication are planted. However, only at the next level, abstraction, do students see relationships between classes of figures (e.g., understand that a square is a rectangle because it has all the properties of rectangles).• Competence at this level affords the learning of higher-level geometry, including deductive arguments and proof.

Thus, learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years.

Composing and decomposing. As with their learning of shapes, components, and properties, students follow a progression to learn about the composition and decomposition of shapes. Initially, they lack competence in composing geometric shapes. With experience, they gain abilities to combine shapes into pictures–first, through trial and error, then gradually using attributes. Finally, they are able to synthesize combinations of shapes into new shapes.•

Students compose new shapes by putting two or more shapes together and discuss the shapes involved as the parts and the totals. They decompose shapes in two ways. They take away a part by covering the total with a part (for example, covering the "top" of a

[1] In this progression, the term "property" is reserved for those attributes that indicate a relationship between components of shapes. Thus, "having parallel sides" or "having all sides of equal lengths" are properties. "Attributes" and "features" are used interchangeably to indicate any characteristic of a shape, including properties, and other defining characteristics (e.g., straight sides) and nondefining characteristics (e.g., "right-side up").

• A shape with straight sides is closed if exactly two sides meet at every vertex, every side meets exactly two other sides, and no sides cross each other.

Levels of geometric thinking

Visual/syncretic. Students recognize shapes, e.g., a rectangle "looks like a door."

Descriptive. Students perceive properties of shapes, e.g., a rectangle has four sides, all its sides are straight, opposite sides have equal length.

Analytic. Students characterize shapes by their properties, e.g., a rectangle has opposite sides of equal length and four right angles.

Abstract. Students understand that a rectangle is a parallelogram because it has all the properties of parallelograms.

• Note that in the U.S., that the term "trapezoid" may have two different meanings. In their study *The Classification of Quadrilaterals* (Information Age Publishing, 2008), Usiskin et al. call these the exclusive and inclusive definitions:

T(E): a trapezoid is a quadrilateral with exactly one pair of parallel sides

T(I): a trapezoid is a quadrilateral with at least one pair of parallel sides.

These different meanings result in different classifications at the analytic level. According to T(E), a parallelogram is not a trapezoid; according to T(I), a parallelogram is a trapezoid.

Both definitions are legitimate. However, Usiskin et al. conclude, "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-bound geometry books, to favor the inclusive definition."

• **A note about research** The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis. Additionally, there is suggestive evidence that this type of composition corresponds with, and may support, children's ability to compose and decompose numbers.

triangle with a smaller triangle to make a trapezoid). And they take shapes apart by building a copy beside the original shape to see what shapes that shape can be decomposed into (initially, they may need to make the decomposition on top of the total shape). With experience, students are able to use a composed shape as a new unit in making other shapes. Grade 1 students make and use such a unit of units (for example, making a square or a rectangle from two identical right triangles, then making pictures or patterns with such squares or rectangles). Grade 2 students make and use three levels of units (making an isosceles triangle from two 1″ by 2″ right triangles, then making a rhombus from two of such isosceles triangles, and then using such a rhombus with other shapes to make a picture or a pattern). Grade 2 students also compose with two such units of units (for example, making adjacent strips from a shorter parallelogram made from a 1″ by 2″ rectangle and two right triangles and a longer parallelogram made from a 1″ by 3″ parallelogram and the same two right triangles). Grade 1 students also rearrange a composite shape to make a related shape, for example, they change a 1″ by 2″ rectangle made from two right triangles into an isosceles triangle by flipping one right triangle. They explore such rearrangements of the two right triangles more systematically by matching the short right angle side (a tall isosceles triangle and a parallelogram with a "little slant"), then the long right angle sides (a short isosceles triangle and a parallelogram with a "long slant"). Grade 2 students rearrange more complex shapes, for example, changing a parallelogram made from a rectangle and two right triangles into a trapezoid by flipping one of the right triangles to make a longer and a shorter parallel side.

Composing and decomposing requires and thus builds experience with properties such as having equal lengths or equal angles.

Spatial structuring and spatial relations. Early composition and decomposition of shape is a foundation for spatial structuring, an important case of geometric composition and decomposition. Students need to conceptually structure an array to understand two-dimensional objects and sets of such objects in two-dimensional space as truly two-dimensional. Such spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because it takes previously abstracted items as content and integrates them to form new structures. For two-dimensional arrays, students must see a composite of squares (iterated units) and as a composite of rows or columns (units of units). Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane. Spatial relations such as above/below and right/left are understood within such spatial structures. These understandings begin informally, later becoming more formal.

The ability to structure a two-dimensional rectangular region into rows and columns of squares requires extended experiences with shapes derived from squares (e.g., squares, rectangles, and right triangles) and with arrays of contiguous squares that form patterns. Development of this ability benefits from experience with compositions, decompositions, and iterations of the two, but it requires extensive experience with arrays.

Students make pictures from shapes whose sides or points touch, and they fill in outline puzzles. These gradually become more elaborate, and students build mental visualizations that enable them to move from trial and error rotating of a shape to planning the orientation and moving the shape as it moves toward the target location. Rows and columns are important units of units within square arrays for the initial study of area, and squares of 1 by 1, 1 by 10, and 10 by 10 are the units, units of units, and units of units of units used in area models of two-digit multiplication in Grade 4. Layers of three-dimensional shapes are central for studying volume in Grade 5. Each layer of a right rectangular prism can also be structured in rows and columns, such layers can also be viewed as units of units of units. That is, as 1000 is a unit (one thousand) of units (one hundred) of units (tens) of units (singletons), a right rectangular prism can be considered a unit (solid, or three-dimensional array) of units (layers) of units (rows) of units (unit cubes).

Summary. The Standards for Kindergarten, Grade 1, and Grade 2 focus on three major aspects of geometry. Students build understandings of shapes and their properties, becoming able to do and discuss increasingly elaborate compositions, decompositions, and iterations of the two, as well as spatial structures and relations. In Grade 2, students begin the formal study of measure, learning to use units of length and use and understand rulers. Measurement of angles and parallelism are a focus in Grades 3, 4, and 5. At Grade 3, students begin to consider relationships of shape categories, considering two levels of subcategories (e.g., rectangles are parallelograms and squares are rectangles). They complete this categorization in Grade 5 with all necessary levels of categories and with the understanding that any property of a category also applies to all shapes in any of its subcategories. They understand that some categories overlap (e.g., not all parallelograms are rectangles) and some are disjoint (e.g., no square is a triangle), and they connect these with their understanding of categories and subcategories. Spatial structuring for two- and three-dimensional regions is used to understand what it means to measure area and volume of the simplest shapes in those dimensions: rectangles at Grade 3 and right rectangular prisms at Grade 5 (see the Geometric Measurement Progression).

K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

Kindergarten

Understanding and describing shapes and space is one of the two critical areas of Kindergarten mathematics. Students develop geometric concepts and spatial reasoning from experience with two perspectives on space: the shapes of objects and the relative positions of objects.

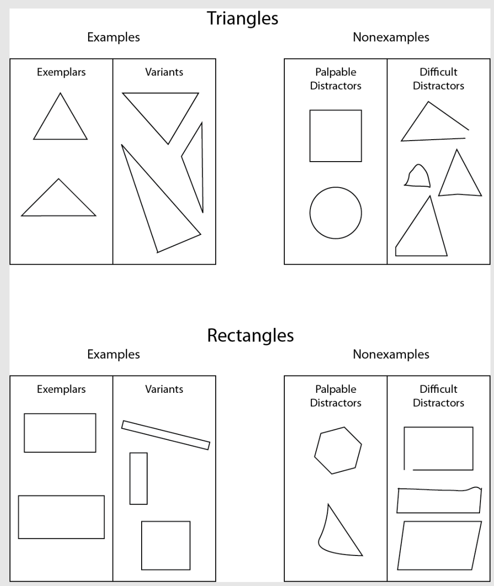
In the domain of shape, students learn to match two-dimensional shapes even when the shapes have different orientations.[K.G.4] They learn to name shapes such as circles, triangles, and squares, whose names occur in everyday language, and distinguish them from nonexamples of these categories, often based initially on visual prototypes. For example, they can distinguish the most typical examples of triangles from the obvious nonexamples.

From experiences with varied examples of these shapes (e.g., the variants shown in the margin), students extend their initial intuitions to increasingly comprehensive and accurate intuitive concept images of each shape category.• These richer concept images support students' ability to perceive a variety of shapes in their environments and describe these shapes in their own words.[MP7] This includes recognizing and informally naming three-dimensional shapes, e.g., "balls," "boxes," "cans." Such learning might also occur in the context of solving problems that arise in construction of block buildings and in drawing pictures, simple maps, and so forth.

Students then refine their informal language by learning mathematical concepts and vocabulary so as to increasingly describe their physical world from geometric perspectives, e.g., shape, orientation, spatial relations (MP4). They increase their knowledge of a variety of shapes, including circles, triangles, squares, rectangles, and special cases of other shapes such as regular hexagons, and trapezoids with unequal bases and non-parallel sides of equal length.[K.G.1] • They learn to sort shapes according to these categories.[MP7] The need to explain their decisions about shape names or classifications prompts students to attend to and describe certain features of the shapes.[K.G.4] That is, concept images and names they have learned for the shapes are the raw material from which they can abstract common features.[MP2] This also supports their learning to represent shapes informally with drawings and by building them from components (e.g., manipulatives such as sticks).[K.G.5] With repeated experiences such as these, students become more precise (MP6). They begin to attend to attributes, such as being a triangle, square, or rectangle, and being *closed* figures with *straight* sides. Similarly, they attend to the lengths of sides and, in simple situations, the size of angles when comparing shapes.

Students also begin to name and describe three-dimensional shapes with mathematical vocabulary, such as "sphere," "cube," "cylinder," and "cone."[K.G.1] They identify faces of three-dimensional shapes as two-dimensional geometric figures[K.G.4] and explicitly identify shapes as two-dimensional ("flat" or lying in a plane) or three-dimensional

K.G.4 Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).



Exemplars are the typical visual prototypes of the shape category.

Variants are other examples of the shape category.

Palpable distractors are nonexamples with little or no overall resemblance to the exemplars.

Difficult distractors are visually similar to examples but lack at least one defining attribute.

• Tall and Vinner describe *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (See "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics*, *12*, pp. 151–169.) This term was formulated by Shlomo Vinner in 1980.

MP7 Mathematically proficient students look closely to discern a pattern or structure.

K.G.1 Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

• If the exclusive definition of trapezoid is used (see p. 3), such trapezoids would be called isosceles trapezoids.

MP7 Young students, for example, . . . may sort a collection of shapes according to how many sides the shapes have.

MP2 Mathematically proficient students have the ability to abstract a given situation.

K.G.5 Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

("solid").[K.G.3]

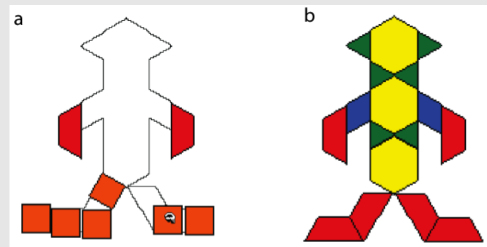
A second important area for kindergartners is the composition of geometric figures. Students not only build shapes from components, but also compose shapes to build pictures and designs. Initially lacking competence in composing geometric shapes, they gain abilities to combine shapes–first by trial and error and gradually by considering components–into pictures. At first, side length is the only component considered. Later experience brings an intuitive appreciation of angle size.

Students combine two-dimensional shapes and solve problems such as deciding which piece will fit into a space in a puzzle, intuitively using geometric motions (slides, flips, and turns, the informal names for translations, reflections, and rotations, respectively). They can construct their own outline puzzles and exchange them, solving each other's.

Finally, in the domain of spatial reasoning, students discuss not only shape and orientation, but also the relative positions of objects, using terms such as "above," "below," "next to," "behind," "in front of," and "beside."[K.G.1] They use these spatial reasoning competencies, along with their growing knowledge of three-dimensional shapes and their ability to compose them, to model objects in their environment, e.g., building a simple representation of the classroom using unit blocks and/or other solids (MP4).

[K.G.3] Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Combining shapes to build pictures



Students first use trial and error (part a) and gradually consider components (part b).

Grade 1

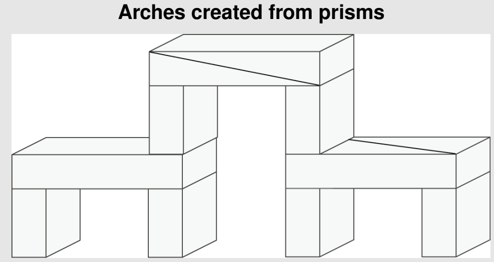
In Grade 1, students reason about shapes. They describe and classify shapes, including drawings, manipulatives, and physical-world objects, in terms of their geometric *attributes*. That is, based on early work recognizing, naming, sorting, and building shapes from components, they describe in their own words why a shape belongs to a given category, such as squares, triangles, circles, rectangles, rhombuses, (regular) hexagons, and trapezoids (with bases of different lengths and nonparallel sides of the same length). In doing so, they differentiate between geometrically defining attributes (e.g., "hexagons have six straight sides") and nondefining attributes (e.g., color, overall size, or orientation).[1.G.1] For example, they might say of this shape, "This has to go with the squares, because all four sides are the same, and these are square corners. It doesn't matter which way it's turned" (MP3, MP7). They explain why the variants shown earlier (p. 6) are members of familiar shape categories and why the difficult distractors are not, and they draw examples and nonexamples of the shape categories. Students learn to sort shapes accurately and exhaustively based on these attributes, describing the similarities and differences of these familiar shapes and shape categories (MP7, MP8).

[1.G.1] Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) ; build and draw shapes to possess defining attributes.

From the early beginnings of informally matching shapes and solving simple shape puzzles, students learn to intentionally compose and decompose plane and solid figures (e.g., putting two congruent isosceles triangles together with the explicit purpose of making a rhombus),[1.G.2] building understanding of part-whole relationships as well as the properties of the original and composite shapes. In this way, they learn to perceive a combination of shapes as a single new shape (e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles). Thus, they develop competencies that include solving shape puzzles and constructing designs with shapes, creating and maintaining a shape as a unit, and combining shapes to create composite shapes that are conceptualized as independent entities (MP2). They then learn to substitute one composite shape for another congruent composite composed of different parts.

[1.G.2] Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[2]

Arches created from prisms



Right rectangular prisms are composed with prisms with right triangle bases. Note that the dimensions of the triangular prism on the top arch differ from the dimensions of that on the right.

Students build these competencies, often more slowly, in the domain of three-dimensional shapes. For example, students may intentionally combine two right triangular prisms to create a right rectangular prism, and recognize that each triangular prism is half of the rectangular prism.[1.G.3] They also show recognition of the composite shape of "arch." (Note that the process of combining shapes to create a composite shape is much like combining 10 ones to make 1 ten.) Even simple compositions, such as building a floor or wall of rectangular prisms, build a foundation for later mathematics.

[1.G.3] Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

As students combine shapes, they continue to develop their sophistication in describing geometric attributes and properties and determining how shapes are alike and different, building foundations for measurement and initial understandings of properties such as congruence and symmetry. Students can learn to use their intuitive understandings of measurement, congruence, and symmetry to guide their work on tasks such as solving puzzles and making simple origami constructions by folding paper to make a given two- or three-dimensional shape (MP1).•

• For example, students might fold a square of paper once to make a triangle or nonsquare rectangle. For examples of other simple two- and three-dimensional origami constructions, see http://www.origami-instructions.com/simple-origami.html.

Grade 2

Students learn to name and describe the defining attributes of categories of two-dimensional shapes, including circles, triangles, squares, rectangles, rhombuses, trapezoids, and the general category of quadrilateral. They describe pentagons, hexagons, septagons, octagons, and other polygons by the number of sides, for example, describing a septagon as either a "seven-gon" or simply "seven-sided shape" (MP2).[2.G.1] Because they have developed both verbal descriptions of these categories and their defining attributes and a rich store of associated mental images, they are able to draw shapes with specified attributes, such as a shape with five sides or a shape with six angles.[2.G.1] They can represent these shapes' attributes accurately (within the constraints of fine motor skills). They use length to identify the properties of shapes (e.g., a specific figure is a rhombus because all four of its sides have equal length). They recognize right angles, and can explain the distinction between a rectangle and a parallelogram without right angles and with sides of different lengths (sometimes called a "rhomboid").

2.G.1 Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[3] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

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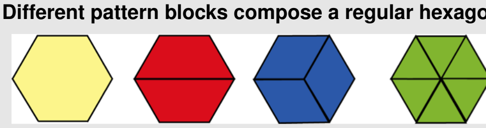
Students learn to combine their composition and decomposition competencies to build and operate on composite units (units of units), intentionally substituting arrangements or composites of smaller shapes or substituting several larger shapes for many smaller shapes, using geometric knowledge and spatial reasoning to develop foundations for area, fraction, and proportion. For example, they build the same shape from different parts, e.g., making with pattern blocks, a regular hexagon from two trapezoids, three rhombuses, or six equilateral triangles. They recognize that the hexagonal faces of these constructions have equal area, that each trapezoid has half of that area, and each rhombus has a third of that area.[2.G.3]

Different pattern blocks compose a regular hexagon

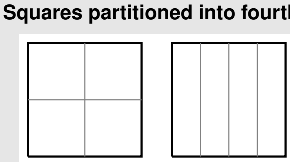


2.G.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

This example emphasizes the fraction concepts that are developed; students can build and recognize more difficult composite shapes and solve puzzles with numerous pieces. For example, a tangram is a special set of 7 shapes which compose an isosceles right triangle. The tangram pieces can be used to make many different configurations and tangram puzzles are often posed by showing pictures of these configurations as silhouettes or outlines. These pictures often are made more difficult by orienting the shapes so that the sides of right angles are not parallel to the edges of the page on which they are displayed. Such pictures often do not show a grid that shows the composing shapes and are generally not preceded by analysis of the composing shapes.

Students also explore decompositions of shapes into regions that are congruent or have equal area.[2.G.3] For example, two squares can be partitioned into fourths in different ways. Any of these fourths represents an equal share of the shape (e.g., "the same amount of cake") even though they have different shapes.

Squares partitioned into fourths



These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

2.G.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Another type of composition and decomposition is essential to students' mathematical development—*spatial structuring*. Students need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP7).

Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because previously abstracted items (e.g., squares, including composites made up of squares) are used as the content of new mental structures. Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. At first, students might tile a rectangle with identical squares or draw such arrays and then count the number of squares one-by-one. In the lowest levels of the progression, they may even lose count of or double-count some squares. As the mental structuring process helps them organize their counting, they become more systematic, using the array structure to guide the quantification. Eventually, they begin to use repeated addition of the number in each row or each column. Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane.

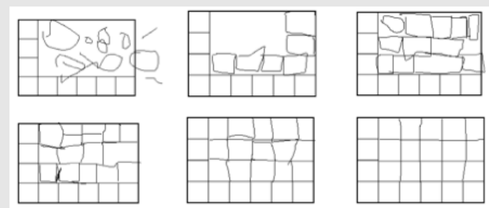
Foundational activities, such as forming arrays by tiling a rectangle with identical squares (as in building a floor or wall from blocks) should have developed students' basic spatial structuring competencies before second grade—if not, teachers should ensure that their students learn these skills. Spatial structuring can be further developed with several activities with grids. Games such as "battleship" can be useful in this regard.

Another useful type of instructional activity is copying and creating designs on grids. Students can copy designs drawn on grid paper by placing manipulative squares and right triangles onto other copies of the grid. They can also create their own designs, draw their creations on grid paper, and exchange them, copying each others' designs.

Another, more complex, activity designing tessellations by iterating a "core square." Students design a unit composed of smaller units: a core square composed of a 2 by 2 array of squares filled with square or right triangular regions. They then create the tessellation ("quilt") by iterating that core in the plane. This builds spatial structuring because students are iterating "units of units" and reflecting on the resulting structures. Computer software can

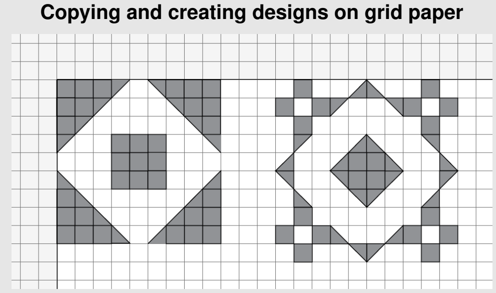
2.G.2 Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Levels of thinking in spatial structuring



Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

Copying and creating designs on grid paper

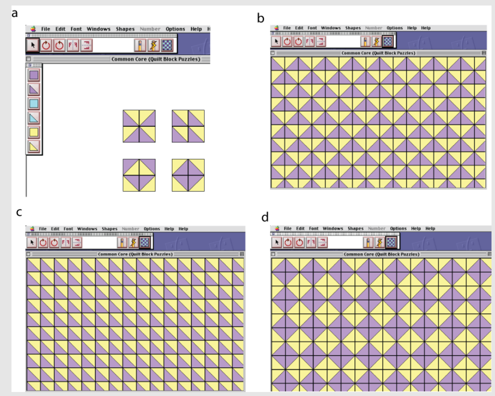


Students can copy designs such as these, using only squares (all of the same size) and isosceles right triangles (half of the square) as manipulatives, creating their copies on paper with grid squares of the same size as the manipulative square.

aid in this iteration.

These various types of composition and decomposition experiences simultaneously develop students' visualization skills, including recognizing, applying, and anticipating (MP1) the effects of applying rigid motions (slides, flips, and turns) to two-dimensional shapes.

"Core squares" iterated to make a tessellation



In the software environment illustrated above (Pattern Blocks and Mini-Quilts software), students need to be explicitly aware of the transformations they are using in order to use slide, flip, and turn tools. At any time, they can tessellate any one of the core squares using the "quilt" tool indicated by the rightmost icon. Part a shows four different core squares. The upper left core square produces the tessellation in part b. Parts c and d are produced, respectively, by the upper right and lower right core squares. Interesting discussions result when the class asks which designs are mathematically different (e.g., should a rotation or flip of the core be counted as "different"s?).

Grade 3

Students analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3).[3.G.1] They explicitly relate and combine these classifications. Because they have built a firm foundation of several shape categories, these categories can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories. A description of these categories of quadrilaterals is illustrated in the margin. The Standards do not require that such representations be constructed by Grade 3 students, but they should be able to draw examples of quadrilaterals that are not in the subcategories.

Similarly, students learn to draw shapes with prespecified attributes, without making a priori assumptions regarding their classification.[MP1] For example, they could solve the problem of making a shape with two long sides of the same length and two short sides of the same length that is not a rectangle.

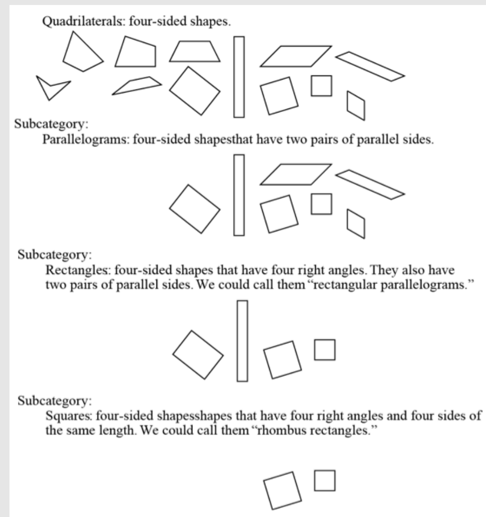
Students investigate, describe, and reason about decomposing and composing polygons to make other polygons. Problems such as finding all the possible different compositions of a set of shapes involve geometric problem solving and notions of congruence and symmetry (MP7). They also involve the practices of making and testing conjectures (MP1), and convincing others that conjectures are correct (or not) (MP3). Such problems can be posed even for sets of simple shapes such as tetrominoes, four squares arranged to form a shape so that every square shares at least one side and sides coincide or share only a vertex.

More advanced paper-folding (origami) tasks afford the same mathematical practices of seeing and using structure, conjecturing, and justifying conjectures. Paper folding can also illustrate many geometric concepts. For example, folding a piece of paper creates a line segment. Folding a square of paper twice, horizontal edge to horizontal edge, then vertical edge to vertical edge, creates a right angle, which can be unfolded to show four right angles. Students can be challenged to find ways to fold paper into rectangles or squares and to explain why the shapes belong in those categories.

Students also develop more competence in the composition and decomposition of rectangular regions, that is, spatially structuring rectangular arrays. They learn to partition a rectangle into identical squares[3.G.2] by anticipating the final structure and thus forming the array by drawing rows and columns (see the bottom right example on p. 11; some students may still need work building or drawing squares inside the rectangle first). They count by the number of columns or rows, or use multiplication to determine the number of

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

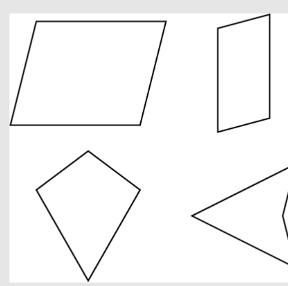
Quadrilaterals and some special kinds of quadrilaterals



The representations above might be used by teachers in class. Note that the left-most four shapes in the first section at the top left have four sides but do not have properties that would place them in any of the other categories shown (parallelograms, rectangles, squares).

[MP1] Students . . . analyze givens, constraints, relationships, and goals.

Quadrilaterals that are not rectangles



These quadrilaterals have two pairs of sides of the same length but are not rectangles. A kite is on lower left and a deltoid is at lower right.

[3.G.2] Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

squares in the array. They also learn to rotate these arrays physically and mentally to view them as composed of smaller arrays, allowing illustrations of properties of multiplication (e.g., the commutative property and the distributive property).

Grade 4

Students describe, analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 3), including explicit use of angle sizes[4.G.1] and the related geometric properties of perpendicularity and parallelism.[4.G.2] They can identify these properties in two-dimensional figures. They can use side length to classify triangles as equilateral, equiangular, isosceles, or scalene; and can use angle size to classify them as acute, right, or obtuse. They then learn to cross-classify, for example, naming a shape as a right isosceles triangle. Thus, students develop explicit awareness of and vocabulary for many concepts they have been developing, including points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Such mathematical terms are useful in communicating geometric ideas, but more important is that constructing examples of these concepts, such as drawing angles and triangles that are acute, obtuse, and right,[4.G.1] help students form richer concept images connected to verbal definitions. That is, students have more complete and accurate mental images and associated vocabulary for geometric ideas (e.g., they understand that angles can be larger than 90° and their concept images for angles include many images of such obtuse angles). Similarly, students see points and lines as abstract objects: Lines are infinite in extent and points have location but no dimension. Grids are made of points and lines and do not end at the edge of the paper.

Students also learn to apply these concepts in varied contexts (MP4). For example, they learn to represent angles that occur in various contexts as two rays, explicitly including the reference line, e.g., a horizontal or vertical line when considering slope or a "line of sight" in turn contexts. They understand the size of the angle as a rotation of a ray on the reference line to a line depicting slope or as the "line of sight" in computer environments. Students might solve problems of drawing shapes with turtle geometry.• Analyzing the shapes in order to construct them (MP1) requires students to explicitly formulate their ideas about the shapes (MP4, MP6). For instance, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such experiences help students connect what are often initially isolated ideas about the concept of angle.

Students might explore line segments, lengths, perpendicularity, and parallelism on different types of grids, such as rectangular and triangular (isometric) grids (MP1, MP2).[4.G.2, 4.G.3] Can you find a non-rectangular parallelogram on a rectangular grid? Can you find a rectangle on a triangular grid? Given a segment on a rectangular grid that is not parallel to a grid line, draw a parallel segment of the same length with a given endpoint. Given a half of a figure and

[4.G.1] Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

[4.G.2] Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

[4.G.1] Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

• The computer programming language Logo has a pointer, often a icon of a turtle, that draws representations of points, line segments, and shapes, with commands such as "forward 100" and "right 120."

[4.G.2] Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

[4.G.3] Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

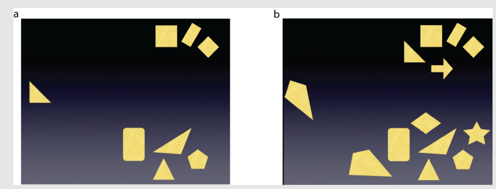
a line of symmetry, can you accurately draw the other half to create a symmetric figure?

Students also learn to reason about these concepts. For example, in "guess my rule" activities, they may be shown two sets of shapes and asked where a new shape belongs (MP1, MP2).[4.G.2]

In an interdisciplinary lesson (that includes science and engineering ideas as well as items from mathematics), students might encounter another property that all triangles have: rigidity. If four fingers (both thumbs and index fingers) form a shape (keeping the fingers all straight), the shape of that quadrilateral can be easily changed by changing the angles. However, using three fingers (e.g., a thumb on one hand and the index and third finger of the other hand), students can see that the shape is fixed by the side lengths. Triangle rigidity explains why this shape is found so frequently in bridge, high-wire towers, amusement park rides, and other constructions where stability is sought.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

Guess My Rule



Students can be shown the two groups of shapes in part a and asked "Where does the shape on the left belong?" They might surmise that it belongs with the other triangles at the bottom. When the teacher moves it to the top, students must search for a different rule that fits all the cases.

Later (part b), students may induce the rule: "Shapes with at least one right angle are at the top." Students with rich visual images of right angles and good visualization skills would conclude that the shape at the left (even though it looks vaguely like another one already at the bottom) has one right angle, thus belongs at the top.

Grade 5

By the end of Grade 5, competencies in shape composition and decomposition, and especially the special case of spatial structuring of rectangular arrays (recall p. 11), should be highly developed (MP7). Students need to develop these competencies because they form a foundation for understanding multiplication, area, volume, and the coordinate plane. To solve area problems, for example, the ability to decompose and compose shapes plays multiple roles. First, students understand that the area of a shape (in square units) is the number of unit squares it takes to cover the shape without gaps or overlaps. They also use decomposition in other ways. For example, to calculate the area of an "L-shaped" region, students might decompose the region into rectangular regions, then decompose each region into an array of unit squares, spatially structuring each array into rows or columns. Students extend their spatial structuring in two ways. They learn to spatially structure in three dimensions; for example, they can decompose a right rectangular prism built from cubes into layers, seeing each layer as an array of cubes. They use this understanding to find the volumes of right rectangular prisms with edges whose lengths are whole numbers as the number of unit cubes that pack the prisms (see the Geometric Measurement Progression). Second, students extend their knowledge of the coordinate plane, understanding the continuous nature of two-dimensional space and the role of fractions in specifying locations in that space.

Thus, spatial structuring underlies coordinates for the plane as well, and students learn both to apply it and to distinguish the objects that are structured. For example, they learn to interpret the components of a rectangular grid structure as line segments or lines (rather than regions) and understand the precision of location that these lines require, rather than treating them as fuzzy boundaries or indicators of intervals. Students learn to reconstruct the levels of counting and quantification that they had already constructed in the domain of discrete objects to the coordination of (at first) two continuous linear measures. That is, they learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions.[5.G.1]

5.G.1 Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).

Although students can often "locate a point," these understandings are beyond simple skills. For example, initially, students often fail to distinguish between two different ways of viewing the point $(2, 3)$, say, as instructions: "right 2, up 3"; and as the point defined by being a distance 2 from the y-axis and a distance 3 from the x-axis. In these two descriptions the 2 is first associated with the x-axis, then with the y-axis.

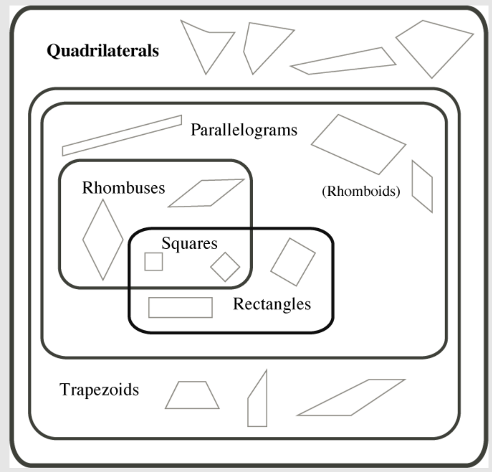
They connect ordered pairs of (whole number) coordinates to points on the grid, so that these coordinate pairs constitute numerical objects and ultimately can be operated upon as single mathematical entities. Students solve mathematical and real-world problems using coordinates. For example, they plan to draw a symmetric figure using computer software in which students' input coordinates that are then connected by line segments.[5.G.2]

Students learn to analyze and relate categories of two-dimensional and three-dimensional shapes explicitly based on their properties.[5.G.4] Based on analysis of properties, they classify two-dimensional figures in hierarchies. For example, they conclude that all rectangles are parallelograms, because they are all quadrilaterals with two pairs of opposite, parallel, equal-length sides (MP3). In this way, they relate certain categories of shapes as subclasses of other categories.[5.G.3] This leads to understanding propagation of properties; for example, students understand that squares possess all properties of rhombuses and of rectangles. Therefore, if they then show that rhombuses' diagonals are perpendicular bisectors of one another, they infer that squares' diagonals are perpendicular bisectors of one another as well.

5.G.2 Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

5.G.4 Classify two-dimensional figures in a hierarchy based on properties.

Venn diagram showing classification of quadrilaterals



Note that rhomboids are parallelograms that are not rhombuses or rectangles. This example uses the inclusive definition of trapezoid (see p. [pageref "T(E)"]).

5.G.3 Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

Grade 6

Problems involving areas and volumes extend previous work and provide a context for developing and using equations.[6.G.1, 6.G.2] Students' competencies in shape composition and decomposition, especially with spatial structuring of rectangular arrays (recall p. 11), should be highly developed. These competencies form a foundation for understanding multiplication, formulas for area and volume, and the coordinate plane.[6.NS.6, 6.NS.8]

Using the shape composition and decomposition skills acquired in earlier grades, students learn to develop area formulas for parallelograms, then triangles. They learn how to address three different cases for triangles: a height that is a side of a right angle, a height that "lies over the base" and a height that is outside the triangle.[MP.1]

Through such activity, students learn that that any side of a triangle can be considered as a base and the choice of base determines the height (thus, the base is not necessarily horizontal and the height is not always in the interior of the triangle). The ability to view a triangle as part of a parallelogram composed of two copies of that triangle and the understanding that area is additive (see the Geometric Measurement Progression) provides a justification (MP3) for halving the product of the base times the height, helping students guard against the common error of forgetting to take half.

Also building on their knowledge of composition and decomposition, students decompose rectilinear polygons into rectangles, and decompose special quadrilaterals and other polygons into triangles and other shapes, using such decompositions to determine their areas, and justifying and finding relationships among the formulas for the areas of different polygons.

Building on the knowledge of volume (see the Geometric Measurement Progression) and spatial structuring abilities developed in earlier grades, students learn to find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism.[6.G.2 MP.1 MP.4]

Students also analyze and compose and decompose polyhedral solids. They describe the shapes of the faces, as well as the number of faces, edges, and vertices. They make and use drawings of solid shapes and learn that solid shapes have an outer surface as well as an interior. They develop visualization skills connected to their mathematical concepts as they recognize the existence of, and visualize, components of three-dimensional shapes that are not visible from a given viewpoint (MP1). They measure the attributes of these shapes, allowing them to apply area formulas to solve surface area problems (MP7). They solve problems that require them to distinguish between units used to measure volume and units used to measure area (or length). They learn to plan the construction of

6.G.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

6.NS.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

MP.1 Students . . . try special cases and simpler forms of the original problem in order to gain insight into its solution.

6.G.2 Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

MP.1 explain correspondences

MP.4 write an equation to describe a situation.

complex three-dimensional compositions through the creation of corresponding two-dimensional nets (e.g., through a process of digital fabrication and/or graph paper).[6.G.4] For example, they may design a living quarters (e.g., a space station) consistent with given specifications for surface area and volume (MP2, MP7). In this and many other contexts, students learn to apply these strategies and formulas for areas and volumes to the solution of real-world and mathematical problems.[6.G.1, 6.G.2] These problems include those in which areas or volumes are to be found from lengths or lengths are to be found from volumes or areas and lengths.

Students extend their understanding of properties of two-dimensional shapes to use of coordinate systems.[6.G.3] For example, they may specify coordinates for a polygon with specific properties, justifying the attribution of those properties through reference to relationships among the coordinates (e.g., justifying that a shape is a parallelogram by computing the lengths of its pairs of horizontal and vertical sides).

As a precursor for learning to describe cross-sections of three-dimensional figures,[7.G.3] students use drawings and physical models to learn to identify parallel lines in three-dimensional shapes, as well as lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.

[6.G.4] Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

[6.G.1] Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[6.G.2] Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

[6.G.3] Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

[7.G.3] Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Where the Geometry Progression is Heading

Composition and decomposition of shapes is used throughout geometry from Grade 6 to high school and beyond. Compositions and decompositions of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems that involve complex shapes. Decompositions are often indicated in geometric diagrams by an auxiliary line, and using the strategy of drawing an auxiliary line to solve a problem are part of looking for and making use of structure (MP7). Recognizing the significance of an existing line in a figure is also part of looking for and making use of structure. This may involve identifying the length of an associated line segment, which in turn may rely on students' abilities to identify relationships of line segments and angles in the figure. These abilities become more sophisticated as students gain more experience in geometry. In Grade 7, this experience includes making scale drawings of geometric figures and solving problems involving angle measure, surface area, and volume (which builds on understandings described in the Geometric Measurement Progression as well as the ability to compose and decompose figures).

Progressions for the Common Core State Standards in Mathematics (draft)

19 September 2013

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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

Chapter 1

Number and Operations—Fractions, 3–5

Overview

Overview to be written.

Note. Changes such as including relevant equations or replacing with tape diagrams or fraction strips are planned for some diagrams. Some readers may find it helpful to create their own equations or representations.

Grade 3

The meaning of fractions In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.[2.G.3] In Grade 3 they start to develop the idea of a fraction more formally, building on the idea of partitioning a whole into equal parts. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision and measurement. In Grade 4, this is extended to include wholes that are collections of objects.

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is the quantity you get by putting 3 of the $\frac{1}{4}$'s together.[3.NF.1] They read any fraction this way, and in particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is the quantity you get by combining 5 parts together when the whole is divided into 3 equal parts.

[3.NF.1] Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

The importance of specifying the whole

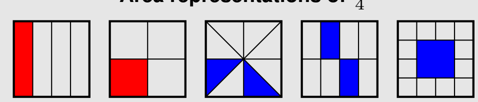
Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP6):

- Specifying the whole.
- Explaining what is meant by "equal parts."

Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Area representations of $\frac{1}{4}$



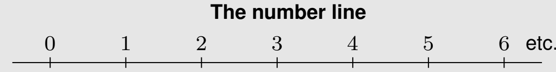
In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3).

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

The number line and number line diagrams On the number line, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler.

The number line

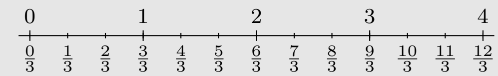


To construct a unit fraction on a number line diagram, e.g. $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They locate the number $\frac{1}{3}$ on the number line by marking off this length from 0, and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.[3.NF.2]

Students sometimes have difficulty perceiving the unit on a number line diagram. When locating a fraction on a number line diagram, they might use as the unit the entire portion of the number line that is shown on the diagram, for example indicating the number 3 when asked to show $\frac{3}{4}$ on a number line diagram marked from 0 to 4. Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, in the early stages of the NF Progression they use other representations such as area models, tape diagrams, and strips of paper. These, like number line diagrams, can be subdivided, representing an important aspect of fractions.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the basic unit of length, namely the interval from 0 to $\frac{1}{3}$.

The number line marked off in thirds

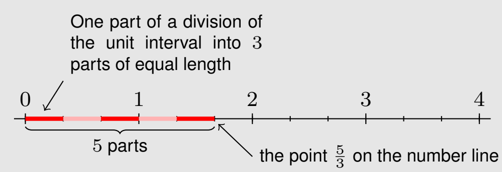


3.NF.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Number line representation of $\frac{5}{3}$



Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions, in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are therefore equal; that is, they are *equivalent fractions*. For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$ and also to $\frac{3}{6}$. Students can also use fraction strips to see fraction equivalence.[3.NF.3ab]

In particular, students in Grade 3 see whole numbers as fractions, recognizing, for example, that the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc. so that[3.NF.3c]

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots .$$

Of particular importance are the ways of writing 1 as a fraction:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots .$$

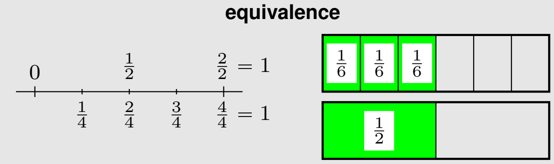
3.NF.3abc Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

Using the number line and fraction strips to see fraction equivalence



Comparing fractions Previously, in Grade 2, students compared lengths using a standard measurement unit.[2.MD.3] In Grade 3 they build on this idea to compare fractions with the same denominator. They see that for fractions that have the same denominator, the underlying unit fractions are the same size, so the fraction with the greater numerator is greater because it is made of more unit fractions. For example, segment from 0 to $\frac{3}{4}$ is shorter than the segment from 0 to $\frac{5}{4}$ because it measures 3 units of $\frac{1}{4}$ as opposed to 5 units of $\frac{1}{4}$. Therefore $\frac{3}{4} < \frac{5}{4}$.[3.NF.3d]

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

3.NF.3d Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

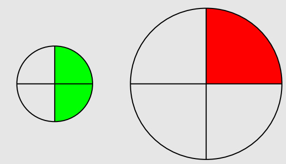
d Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Students also see that for unit fractions, the one with the larger denominator is smaller, by reasoning, for example, that in order for more (identical) pieces to make the same whole, the pieces must be smaller. From this they reason that for fractions that have the same numerator, the fraction with the smaller denominator is greater. For example, $\frac{2}{5} > \frac{2}{7}$, because $\frac{1}{7} < \frac{1}{5}$, so 2 lengths of $\frac{1}{7}$ is less than 2 lengths of $\frac{1}{5}$.

As with equivalence of fractions, it is important in comparing fractions to make sure that each fraction refers to the same whole.

As students move towards thinking of fractions as points on the number line, they develop an understanding of order in terms of position. Given two fractions—thus two points on the number line—the one to the left is said to be smaller and the one to right is said to be larger. This understanding of order as position will become important in Grade 6 when students start working with negative numbers.

The importance of referring to the same whole when comparing fractions



A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

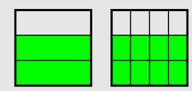
Grade 4

Grade 4 students learn a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction that represents the same number as the original fraction. This property forms the basis for much of their other work in Grade 4, including the comparison, addition, and subtraction of fractions and the introduction of finite decimals.

Equivalent fractions Students can use area models and number line diagrams to reason about equivalence.[4.NF.1] They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, *n*, corresponds physically to partitioning each unit fraction piece into *n* smaller equal pieces. The whole is then partitioned into *n* times as many pieces, and there are *n* times as many smaller unit fraction pieces as in the original fraction.

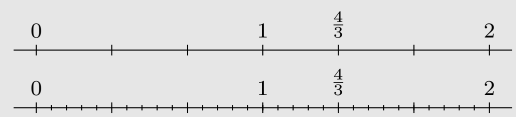
4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area model to show that $\frac{2}{3} = \frac{4\times2}{4\times3}$



The whole is the square, measured by area. On the left it is divided horizontally into 3 rectangles of equal area, and the shaded region is 2 of these and so represents $\frac{2}{3}$. On the right it is divided into 4×3 small rectangles of equal area, and the shaded area comprises 4×2 of these, and so it represents $\frac{4\times2}{4\times3}$.

Using the number line to show that $\frac{4}{3} = \frac{5\times4}{5\times3}$



$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5\times3}$. Divide each of the intervals of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5\times4}{5\times3} = \frac{20}{15}$.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction as a point on the number line.

The fundamental property can be presented in terms of division, as in, e.g.

$$\frac{28}{36} = \frac{28 \div 4}{36 \div 4} = \frac{7}{9}.$$

Because the equations $28 \div 4 = 7$ and $36 \div 4 = 9$ tell us that $28 = 4 \times 7$ and $36 = 4 \times 9$, this is the fundamental fact in disguise:

$$\frac{4 \times 7}{4 \times 9} = \frac{7}{9}.$$

It is possible to over-emphasize the importance of simplifying fractions in this way. There is no mathematical reason why fractions must be written in simplified form, although it may be convenient to do so in some cases.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.[4.NF.2] For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they rewrite both fractions as

$$\frac{60}{96}\left(= \frac{12 \times 5}{12 \times 8}\right) \quad \text{and} \quad \frac{56}{96}\left(= \frac{7 \times 8}{12 \times 8}\right)$$

4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 (and is therefore to the left of 1) but $\frac{13}{12}$ is greater than 1 (and is therefore to the right of 1).

Grade 5 students who have learned about fraction multiplication can see equivalence as "multiplying by 1":

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}$$

However, although a useful mnemonic device, this does not constitute a valid argument at this grade, since students have not yet learned fraction multiplication.

Adding and subtracting fractions The meaning of addition is the same for both fractions and whole numbers, even though algorithms for calculating their sums can be different. Just as the sum of 4 and 7 can be seen as the length of the segment obtained by joining together two segments of lengths 4 and 7, so the sum of $\frac{2}{3}$ and $\frac{8}{5}$ can be seen as the length of the segment obtained joining together two segments of length $\frac{2}{3}$ and $\frac{8}{5}$. It is not necessary to know how much $\frac{2}{3} + \frac{8}{5}$ is exactly in order to know what the sum means. This is analogous to understanding 51×78 as 51 groups of 78, without necessarily knowing its exact value.

This simple understanding of addition as putting together allows students to see in a new light the way fractions are built up from unit fractions. The same representation that students used in Grade 4 to see a fraction as a point on the number line now allows them to see a fraction as a sum of unit fractions: just as $5 = 1+1+1+1+1$, so

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

because $\frac{5}{3}$ is the total length of 5 copies of $\frac{1}{3}$.[4.NF.3]

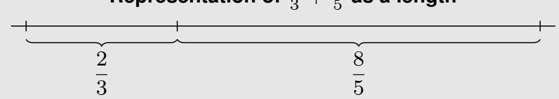
Armed with this insight, students decompose and compose fractions with the same denominator. They add fractions with the same denominator:[4.NF.3c]

$$\begin{aligned}
\frac{7}{5} + \frac{4}{5} &= \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{7} + \overbrace{\frac{1}{5} + \cdots \frac{1}{5}}^{4} \\
&= \frac{\overbrace{1 + 1 + \cdots + 1}^{7+4}}{5} \\
&= \frac{7+4}{5}.
\end{aligned}$$

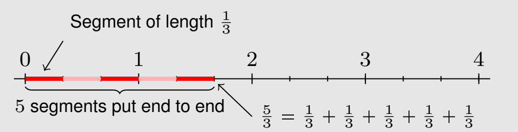
Using the understanding gained from work with whole numbers of the relationship between addition and subtraction, they also subtract fractions with the same denominator. For example, to subtract $\frac{5}{6}$ from $\frac{17}{6}$, they decompose

$$\frac{17}{6} = \frac{12}{6} + \frac{5}{6}, \quad \text{so} \quad \frac{17}{6} - \frac{5}{6} = \frac{17-5}{6} = \frac{12}{6} = 2.$$

Representation of $\frac{2}{3} + \frac{8}{5}$ as a length



Using the number line to see that $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$



4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

$$7\frac{1}{5} = 7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators.• Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as a case of fraction addition.

• A mixed number is a whole number plus a fraction smaller than 1, written without the $+$ sign, e.g. $5\frac{3}{4}$ means $5+\frac{3}{4}$ and $7\frac{1}{5}$ means $7+\frac{1}{5}$.

Similarly, converting an improper fraction to a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.[4.NF.3b] Students can draw on their knowledge from Grade 3 of whole numbers as fractions. For example, knowing that $1 = \frac{3}{3}$, they see

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

4.NF.3b Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

Repeated reasoning with examples that gain in complexity leads to a general method involving the Grade 4 NBT skill of finding quotients and remainders.[4.NBT.6] For example,

$$\frac{47}{6} = \frac{(7 \times 6) + 5}{6} = \frac{7 \times 6}{6} + \frac{5}{6} = 7 + \frac{5}{6} = 7\frac{5}{6}.$$

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

When solving word problems students learn to attend carefully to the underlying unit quantities. In order to formulate an equation of the form $A+B = C$ or $A-B = C$ for a word problem, the numbers A, B, and C must all refer to the same (or equivalent) wholes or unit amounts.[4.NF.3d] For example, students understand that the problem

> Bill had $\frac{2}{3}$ of a cup of juice. He drank $\frac{1}{2}$ of his juice. How much juice did Bill have left?

cannot be solved by subtracting $\frac{2}{3} - \frac{1}{2}$ because the $\frac{2}{3}$ refers to a cup of juice, but the $\frac{1}{2}$ refers to the amount of juice that Bill had, and not to a cup of juice. Similarly, in solving

> If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{3} + \frac{1}{4}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

4.NF.3d Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

d Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 3×7 can be represented as the number of objects in 3 groups of 7 objects, and write this as $7 + 7 + 7$. Grade 4 students apply this understanding to fractions, seeing

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \quad \text{as} \quad 5 \times \frac{1}{3}.$$

In general, they see a fraction as the numerator times the unit fraction with the same denominator,[4.NF.4a] e.g.,

$$\frac{7}{5} = 7 \times \frac{1}{5}, \qquad \frac{11}{3} = 11 \times \frac{1}{3}.$$

The same thinking, based on the analogy between fractions and whole numbers, allows students to give meaning to the product of a whole number and a fraction,[4.NF.4b] e.g., they see

$$3 \times \frac{2}{5} \quad \text{as} \quad \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{3 \times 2}{5} = \frac{6}{5}.$$

Students solve word problems involving multiplication of a fraction by a whole number.[4.NF.4c]

> If a bucket holds $2\frac{3}{4}$ gallons and 43 buckets of water fill a tank, how much does the tank hold?

The answer is $43 \times 2\frac{3}{4}$ gallons, which is

$$43 \times \left(2 + \frac{3}{4}\right) = 43 \times \frac{11}{4} = \frac{473}{4} = 118\frac{1}{4} \text{ gallons}$$

Decimals Fractions with denominator 10 and 100, called *decimal fractions*, arise naturally when student convert between dollars and cents, and have a more fundamental importance, developed in Grade 5, in the base 10 system. For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a concrete context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

Grade 4 students learn to add decimal fractions by converting them to fractions with the same denominator, in preparation for general fraction addition in Grade 5:[4.NF.5]

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents.

Fractions with denominators equal to 10, 100, etc., such

$$\frac{27}{10}, \quad \frac{27}{100}, \quad \text{etc.}$$

can be written by using a *decimal point* as[4.NF.6]

$$2.7, \quad 0.27.$$

The number of digits to the right of the decimal point indicates the number of zeros in the denominator, so that $2.70 = \frac{270}{100}$ and

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

- a Understand a fraction a/b as a multiple of $1/b$.
- b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.
- c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

[4.NF.5] Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[1]

[4.NF.6] Use decimal notation for fractions with denominators 10 or 100.

$2.7 = \frac{27}{10}$. Students use their ability to convert fractions to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

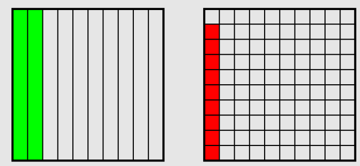
Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of them as 0.20 and 0.09 and see that $0.20 > 0.09$ because[4.NF.7]

$$\frac{20}{100} > \frac{9}{100}.$$

The argument using the meaning of a decimal as a fraction generalizes to work with decimals in Grade 5 that have more than two digits, whereas the argument using a visual fraction model, shown in the margin, does not. So it is useful for Grade 4 students to see such reasoning.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, =, or $<$, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$ using a visual fraction model



The shaded region on the left shows 0.2 of the unit square, since it is two parts when the square is divided int 10 parts of equal area. The shaded region on the right shows 0.09 of the unit square, since it is 9 parts when the unit is divided into 100 parts of equal area.

Grade 5

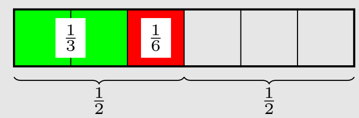
Adding and subtracting fractions In Grade 4, students have some experience calculating sums of fractions with different denominators in their work with decimals, where they add fractions with denominators 10 and 100, such as

$$\frac{2}{10} + \frac{7}{100} = \frac{20}{100} + \frac{7}{100} = \frac{27}{100}.$$

Note that this is a situation where one denominator is a divisor of the other, so that only one fraction has to be changed. They might have encountered other similar situations, for example using a fraction strip to reason that

$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

Using a fraction strip to show that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$



They understand the process as expressing both summands in terms of the same unit fraction so that they can be added. Grade 5 students extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.[5.NF.1] For example, in calculating $\frac{2}{3} + \frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is subdivided into fourths, and if each fourth in $\frac{5}{4}$ is subdivided into thirds, then each fraction will be a sum of unit fractions with denominator $3 \times 4 = 4 \times 3 = 12$:

$$\frac{2}{3} + \frac{5}{4} = \frac{2 \times 4}{3 \times 4} + \frac{5 \times 3}{4 \times 3} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}.$$

5.NF.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

In general two fractions can be added by subdividing the unit fractions in one using the denominator of the other:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{ad + bc}{bd}.$$

It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common denominator is a distraction from understanding algorithms for adding fractions.

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.[5.NF.2] For example in the problem

> Ludmilla and Lazarus each have a lemon. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ a cup from hers and Lazarus squeezes $\frac{2}{5}$ of a cup from his. How much lemon juice to they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5} < \frac{1}{2}$. They calculate $\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$ by noticing that $\frac{3}{7} < \frac{1}{2}$.

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, so that each person gets $50 \times \frac{1}{9} = \frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any quantity by a fraction.[5.NF.4a] Just as

$$\frac{1}{3} \times 5 \text{ is one part when 5 is partitioned into 3 parts,}$$

so

$$\frac{4}{3} \times 5 \text{ is 4 parts when 5 is partitioned into 3 parts.}$$

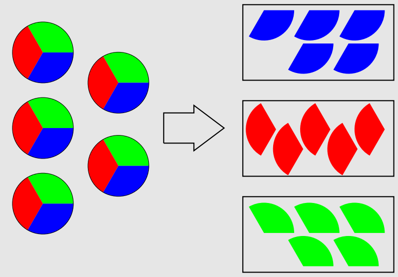
Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:
$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

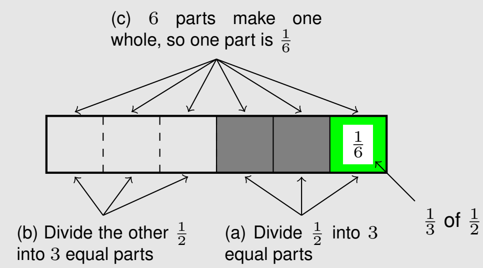


If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

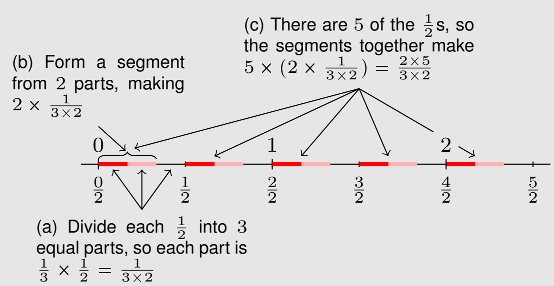
5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2}$



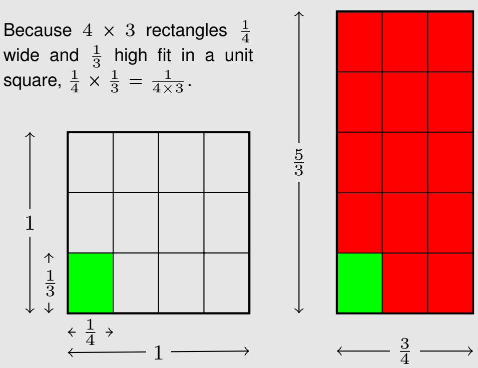
For more complicated examples, an area model is useful, in which students work with a rectangle that has fractional side lengths, dividing it up into rectangles whose sides are the corresponding unit fractions.

5.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3\times 5}{4\times 3}$

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high fit in a unit square, $\frac{1}{4} \times \frac{1}{3} = \frac{1}{4\times 3}$.



The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4\times 3}$, so has area $\frac{3\times 5}{4\times 3}$.

Students also understand fraction multiplication by creating story contexts. For example, to explain

$$\frac{2}{3} \times 4 = \frac{8}{3},$$

they might say

> Ron and Hermione have 4 pounds of Bertie Bott's Every Flavour Beans. They decide to share them 3 ways, saving one share for Harry. How many pounds of beans do Ron and Hermione get?

Using the relationship between division and multiplication, students start working with simple fraction division problems. Having seen that division of a whole number by a whole number, e.g. $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that[5.NF.7a]

$$\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}.$$

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be 3×6 in 3, and so[5.NF.7b]

$$3 \div \frac{1}{6} = 3 \times 6 = 18.$$

Students use story problems to make sense of division problems:[5.NF.7c]

> How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?

5.NF.7abc Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[2]

a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

Students attend carefully to the underlying unit quantities when solving problems. For example, if $\frac{1}{2}$ of a fund-raiser's funds were raised by the 6th grade, and if $\frac{1}{3}$ of the 6th grade's funds were raised by Ms. Wilkin's class, then $\frac{1}{3} \times \frac{1}{2}$ gives the fraction of the fund-raiser's funds that Ms. Wilkin's class raised, but it does not tell us how much money Ms. Wilkin's class raised.[5.NF.6]

5.NF.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

Multiplication as scaling In preparation for Grade 6 work in ratios and proportional reasoning, students learn to see products such as 5×3 or $\frac{1}{2} \times 3$ as expressions that can be interpreted in terms of a quantity, 3, and a scaling factor, 5 or $\frac{1}{2}$. Thus, in addition to knowing that $5 \times 3 = 15$, they can also say that 5×3 is 5 times as big as 3, without evaluating the product. Likewise, they see $\frac{1}{2} \times 3$ as half the size of 3.[5.NF.5a]

The understanding of multiplication as scaling is an important opportunity for students to reasonin abstractly (MP2). Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a recipe is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when the budget of a large state university is multiplied by $\frac{1}{2}$, for example.[5.NF.5b]

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 7.

5.NF.5a Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

5.NF.5b Interpret multiplication as scaling (resizing), by:

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Progressions for the Common Core State Standards in Mathematics (draft)

26 December 2011

6-7, Ratios and Proportional Relationships

Overview

The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades. Ratios and proportional relationships are foundational for further study in mathematics and science and useful in everyday life. Students use ratios in geometry and in algebra when they study similar figures and slopes of lines, and later when they study sine, cosine, tangent, and other trigonometric ratios in high school. Students use ratios when they work with situations involving constant rates of change, and later in calculus when they work with average and instantaneous rates of change of functions. An understanding of ratio is essential in the sciences to make sense of quantities that involve derived attributes such as speed, acceleration, density, surface tension, electric or magnetic field strength, and to understand percentages and ratios used in describing chemical solutions. Ratios and percentages are also useful in many situations in daily life, such as in cooking and in calculating tips, miles per gallon, taxes, and discounts. They also are also involved in a variety of descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g., birth rate, per capita income, body mass index, rain fall, and crop yield) and underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Ratios, rates, proportional relationships, and percent Ratios arise in situations in which two (or more) quantities are related.• Sometimes the quantities have the same units (e.g., 3 cups of apple juice and 2 cups of grape juice), other times they do not (e.g., 3 meters and 2 seconds). Some authors distinguish ratios from rates, using the term "ratio" when units are the same and "rate" when units are different; others use ratio to encompass both kinds of situations. The

• In the Standards, a quantity involves measurement of an attribute. Quantities may be discrete, e.g., 4 apples, or continuous, e.g., 4 inches. They may be measurements of physical attributes such as length, area, volume, weight, or other measurable attributes such as income. Quantities can vary with respect to another quantity. For example, the quantities "distance between the earth and the sun in miles," "distance (in meters) that Sharoya walked," or "my height in feet" vary with time.

Standards use ratio in the second sense, applying it to situations in which units are the same as well as to situations in which units are different. Relationships of two quantities in such situations may be described in terms of ratios, rates, percents, or proportional relationships.

A ratio associates two or more quantities. Ratios can be indicated in words as "3 to 2" and "3 for every 2" and "3 out of every 5" and "3 parts to 2 parts." This use might include units, e.g., "3 cups of flour for every 2 eggs" or "3 meters in 2 seconds." Notation for ratios can include the use of a colon, as in $3:2$. The quotient $\frac{3}{2}$ is sometimes called the value of the ratio $3:2$.•

• In everyday language. the word "ratio" sometimes refers to the value of a ratio, for example in the phrases "take the ratio of price to earnings" or "the ratio of circumference to diameter is π."

Ratios have associated rates. For example, the ratio 3 feet for every 2 seconds has the associated rate $\frac{3}{2}$ feet for every 1 second; the ratio 3 cups apple juice for every 2 cups grape juice has the associated rate $\frac{3}{2}$ cups apple juice for every 1 cup grape juice. In Grades 6 and 7, students describe rates in terms such as "for each 1," "for each," and "per." The *unit rate* is the numerical part of the rate; the "unit" in "unit rate" is often used to highlight the 1 in "for each 1" or "for every 1."

Equivalent ratios arise by multiplying each measurement in a ratio pair by the same positive number. For example, the pairs of numbers of meters and seconds in the margin are in equivalent ratios. Such pairs are also said to be in the same ratio. Proportional relationships involve collections of pairs of measurements in equivalent ratios. In contrast, a proportion is an equation stating that two ratios are equivalent. Equivalent ratios have the same unit rate.

Representing pairs in a proportional relationship

Sharoya walks 3 meters every 2 seconds. Let d be the number of meters Sharoya has walked after t seconds. d and t are in a proportional relationship.

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

d and t are related by the equation $d = \left(\frac{3}{2}\right)t$. Students sometimes use the equals sign incorrectly to indicate proportional relationships, for example, they might write "3 m = 2 sec" to represent the correspondence between 3 meters and 2 seconds. In fact, 3 meters is not equal to 2 seconds. This relationship can be represented in a table or by writing "3 m → 2 sec." Note that the unit rate appears in the pair $\left(\frac{3}{2}, 1\right)$.

The pairs of meters and seconds in the margin show distance and elapsed time varying together in a *proportional relationship*. This situation can be described as "distance traveled and time elapsed are proportionally related," or "distance and time are directly proportional," or simply "distance and time are proportional." The proportional relationship can be represented with the equation $d = \left(\frac{3}{2}\right)t$. The factor $\frac{3}{2}$ is the constant unit rate associated with the different pairs of measurements in the proportional relationship; it is known as a *constant of proportionality*.

The word *percent* means "per 100" (*cent* is an abbreviation of the Latin *centum* "hundred"). If 35 milliliters out of every 100 milliliters in a juice mixture are orange juice, then the juice mixture is 35% orange juice (by volume). If a juice mixture is viewed as made of 100 equal parts, of which 35 are orange juice, then the juice mixture is 35% orange juice.

More precise definitions of the terms presented here and a framework for organizing and relating the concepts are presented in the Appendix.

Recognizing and describing ratios, rates, and proportional relationships "For each," "for every," "per," and similar terms distinguish situations in which two quantities have a proportional relationship from other types of situations. For example, without further information "2 pounds for a dollar" is ambiguous. It may be that pounds and dollars are proportionally related and *every* two pounds costs a dollar. Or it may be that there is a discount on bulk, so weight and cost do not have a proportional relationship. Thus, recognizing ratios, rates, and proportional relationships involves looking for structure (MP7). Describing and interpreting descriptions of ratios, rates, and proportional relationships involves precise use of language (MP6).

Representing ratios, collections of equivalent ratios, rates, and proportional relationships Because ratios and rates are different and rates will often be written using fraction notation in high school, ratio notation should be distinct from fraction notation.

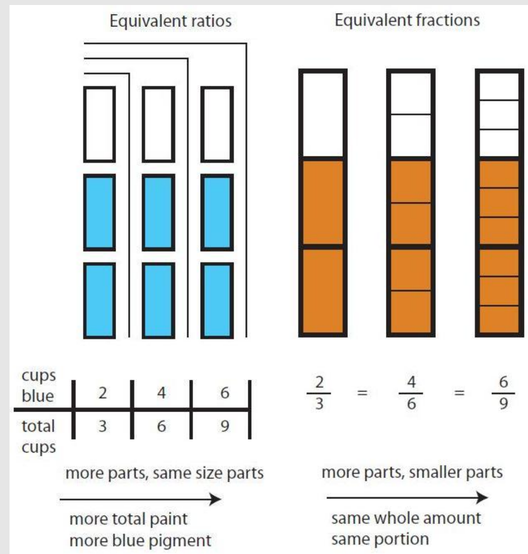
Together with tables, students can also use tape diagrams and double number line diagrams to represent collections of equivalent ratios. Both types of diagrams visually depict the relative sizes of the quantities.

Tape diagrams are best used when the two quantities have the same units. They can be used to solve problems and also to highlight the multiplicative relationship between the quantities.

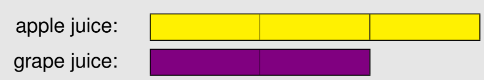
Double number line diagrams are best used when the quantities have different units (otherwise the two diagrams will use different length units to represent the same amount). Double number line diagrams can help make visible that there are many, even infinitely many, pairs in the same ratio, including those with rational number entries. As in tables, unit rates appear paired with 1.

A collection of equivalent ratios can be graphed in the coordinate plane. The graph represents a proportional relationship. The unit rate appears in the equation and graph as the slope of the line, and in the coordinate pair with first coordinate 1.

Equivalent ratios versus equivalent fractions

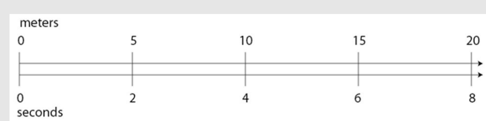


Representing ratios with tape diagrams



This diagram can be interpreted as representing any *mixture of apple juice and grape juice with a ratio of 3 to 2. The total amount of juice is represented as partitioned into 5 parts of equal size, represented by 5 rectangles. For example, if the diagram represents 5 cups of juice mixture, then each of these rectangles represents 1 cup. If the total amount of juice mixture is 1 gallon, then each part represents $\frac{1}{5}$ gallon and there are $\frac{3}{5}$ gallon of apple juice and $\frac{2}{5}$ gallon of grape juice.*

Representing ratios with double number line diagrams



On double number line diagrams, if A and B are in the same ratio, then A and B are located at the same distance from 0 on their respective lines. Multiplying A and B by a positive number p results in a pair of numbers whose distance from 0 is p times as far. So, for example, 3 times the pair 2 and 5 results in the pair 6 and 15 which is located at 3 times the distance from 0.

Grade 6

Representing and reasoning about ratios and collections of equivalent ratios Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole number measurements such as "3 lemons for every \$1" or "for every 5 cups grape juice, mix in 2 cups peach juice" lend themselves to being recorded in a table.[6.RP.3a] Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

6.RP.3a Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.[6.RP.1,6.RP.2] It is important for students to focus on the meaning of the terms "for every," "for each," "for each 1," and "per" because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.[6.EE.9]

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Showing structure in tables and graphs

Additive Structure

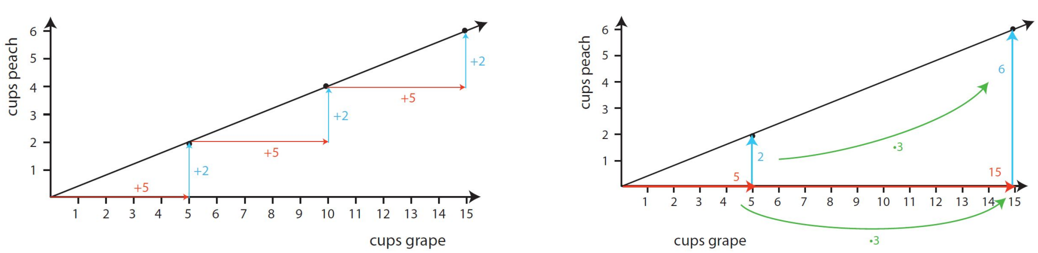
cups grape	cups peach
5	2
10	4
15	6
20	8
25	10

+5 +5 +5 +5 / +2 +2 +2 +2

Multiplicative Structure

cups grape	cups peach
5	2
10	4
15	6
20	8
100	40

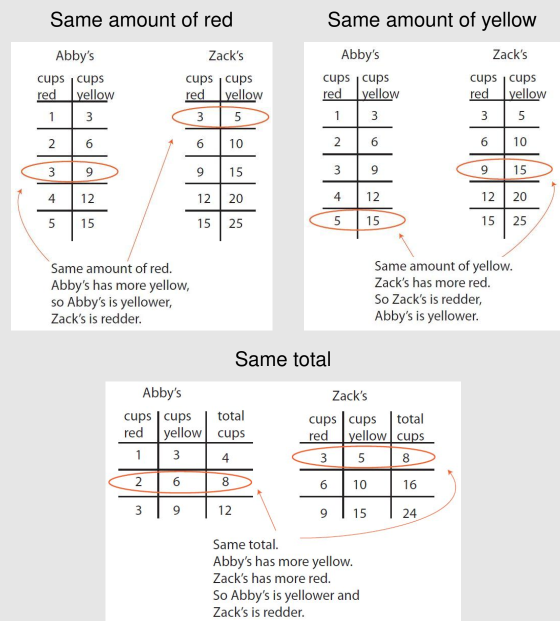
•2 •3 •20 / •2 •3 •20



In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP1).

By reasoning about ratio tables to compare ratios,[6.RP.3a] students can deepen their understanding of what a ratio describes in a context and what quantities in equivalent ratios have in common. For example, suppose Abby's orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint and Zack's orange paint is made by mixing 3 cups red for every 5 cups yellow. Students could discuss that all the mixtures within a single ratio table for one of the paint mixtures are the same shade of orange because they are multiple batches of the same mixture. For example, 2 cups red and 6 cups yellow is two batches of 1 cup red and 3 cups yellow; each batch is the same color, so when the two batches are combined, the shade of orange doesn't change. Therefore, to compare the colors of the two paint mixtures, any entry within a ratio table for one mixture can be compared with any entry from the ratio table for the other mixture.

Three ways to compare paint mixtures

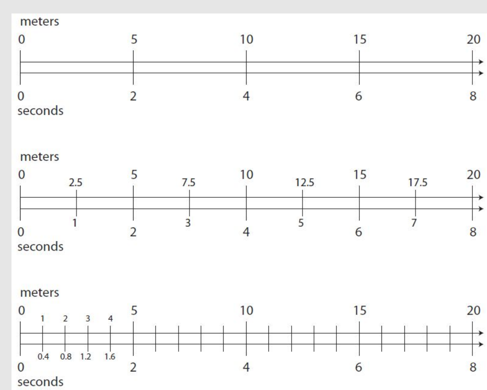


It is important for students to focus on the rows (or columns) of a ratio table as multiples of each other. If this is not done, a common error when working with ratios is to make additive comparisons. For example, students may think incorrectly that the ratios $1 : 3$ and $3 : 5$ of red to yellow in Abby's and Zack's paints are equivalent because the difference between the number of cups of red and yellow in both paints is the same, or because Zack's paint could be made from Abby's by adding 2 cups red and 2 cups yellow. The margin shows several ways students could reason correctly to compare the paint mixtures.

Strategies for solving problems Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6. There are a number of strategies for solving problems that involve ratios. As students become familiar with relationships among equivalent ratios, their strategies become increasingly abbreviated and efficient.

For example, suppose grape juice and peach juice are mixed in a ratio of 5 to 2 and we want to know how many cups of grape juice to mix with 12 cups of peach juice so that the mixture will still be in the same ratio. Students could make a ratio table as shown in the margin, and they could use the table to find the grape juice entry that pairs with 12 cups of peach juice in the table. This perspective allows students to begin to reason about proportions by starting with their knowledge about multiplication tables and by building on this knowledge.

Double number line diagrams used for situations with different units



Double number lines indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.

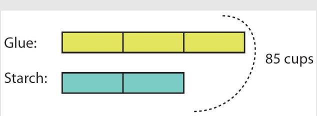
As students generate equivalent ratios and record them in tables, their attention should be drawn to the important role of multiplication and division in how entries are related to each other. Initially, students may fill ratio tables with columns or rows of the multiplication table by skip counting, using only whole number entries, and placing these entries in numerical order. Gradually, students should consider entries in ratio tables beyond those they find by skip counting, including larger entries and fraction or decimal entries. Finding these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N, the distance traveled should also be multiplied (or divided) by N. Double number lines can be useful in representing ratios that involve fractions and decimals.

As students become comfortable with fractional and decimal entries in tables of quantities in equivalent ratios, they should learn to appreciate that unit rates are especially useful for finding entries. A unit rate gives the number of units of one quantity per 1 unit of the other quantity. The amount for N units of the other quantity is then found by multiplying by N. Once students feel comfortable doing so, they may wish to work with abbreviated tables instead of working with long tables that have many values. The most abbreviated tables consist of only two columns or two rows; solving a proportion is a matter of finding one unknown entry in the table.

Measurement conversion provides other opportunities for students to use relationships given by unit rates.[6.RP.3d] For example, recognizing "12 inches in a foot," "1000 grams in a kilogram," or "one kilometer is $\frac{5}{8}$ of a mile" as rates, can help to connect concepts and methods developed for other contexts with measurement conversion.

Representing a problem with a tape diagram

Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?



5 parts $\longrightarrow$ 85 cups
1 part $\longrightarrow$ $85 \div 5 = 17$ cups
3 parts $\longrightarrow$ $3 \cdot 17 = 51$ cups
2 parts $\longrightarrow$ $2 \cdot 17 = 34$ cups

51 cups glue and 34 cups starch are needed.

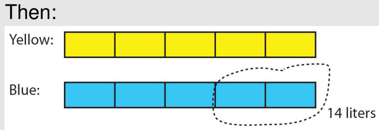
Tape diagrams can be useful aids for solving problems.

Representing a multi-step problem with two pairs of tape diagrams

Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?

At first:

Then:



2 parts $\longrightarrow$ 14 liters
1 part $\longrightarrow$ $14 \div 2 = 7$ liters
(original total) 8 parts $\longrightarrow$ $8 \cdot 7 = 56$ liters

There was 56 liters of green paint to start with.

This problem can be very challenging for sixth or seventh graders.

A progression of strategies for solving a proportion

If 2 pounds of beans cost \$5, how much will 15 pounds of beans cost?

Method 1

pounds	2	4	6	8	10	12	14	1	15
dollars	5	10	15	20	25	30	35	2.50	37.50

"I found 14 pounds costs \$35 and then 1 more pound is another \$2.50, so that makes \$37.50 in all."

Method 2

	$\div 2$	$\cdot 15$	
pounds	2	1	15
dollars	5	2.50	37.50
	$\div 2$	$\cdot 15$	

"I found 1 pound first because if I know how much it costs for each pound then I can find any number of pounds by multiplying."

Method 3

	$\cdot \frac{15}{2}$	
pounds	2	15
dollars	5	37.50
	$\cdot \frac{15}{2}$	

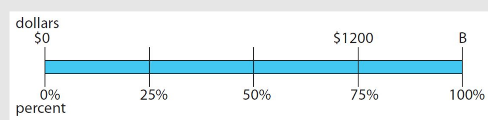
The previous method, done in one step.

With this perspective, the second column is seen as the first column times a number. To solve the proportion one first finds this number.

[6.RP.3d]Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Solving a percent problem

If 75% of the budget is \$1200, what is the full budget?



"I said 75% is 3 parts and is \$1200
25% is 1 part and is $\$1200 \div 3 = \$400$
100% is 4 parts and is $4 \cdot \$400 = \$1600$"

portion	75	3	1200
whole	100	4	1600

75% is $\frac{1200}{B}$ 75% of B is 1200

$\frac{75}{100} = \frac{1200}{B}$ $\frac{75}{100} \cdot B = 1200$

$B = 1600$

In reasoning about and solving percent problems, students can use a variety of strategies. Representations such as this, which is a blend between a tape diagram and a double number line diagram, can support sense-making and reasoning about percent.

Grade 7

In Grade 7, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated unit rates. They identify these unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

At this grade, students will also work with ratios specified by rational numbers, such as $\frac{3}{4}$ cups flour for every $\frac{1}{2}$ stick butter.[7.RP.1] Students continue to use ratio tables, extending this use to finding unit rates.

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

Ratio problem specified by rational numbers: Three approaches

To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you want to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need?

Method 1

		$\cdot 6$	$\cdot 4$
cups blue	$\frac{1}{2}$	3	12
cups red	$\frac{1}{3}$	2	8
total cups purple	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	5	20
		$\cdot 6$	$\cdot 4$

"I thought about making 6 batches of purple because that is a whole number of cups of purple. To make 6 batches, I need 6 times as much blue and 6 times as much red too. That was 3 cups blue and 2 cups red and that made 5 cups purple. Then 4 times as much of each makes 20 cups purple."

Method 2

$$\frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10} \qquad \frac{6}{10} \cdot 20 = 12$$

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15} \qquad \frac{6}{15} \cdot 20 = 8$$

"I found out what fraction of the paint is blue and what fraction is red. Then I found those fractions of 20 to find the number of cups of blue and red in 20 cups."

Method 3

cups blue	$\frac{1}{2}$		12
cups red	$\frac{1}{3}$	$\cdot\frac{3}{5}$	8
total cups purple	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$	$\cdot\frac{2}{5}$	20

Like Method 2, but in tabular form, and viewed as multiplicative comparisons.

Recognizing proportional relationships Students examine situations carefully, to determine if they describe a proportional relationship.[7.RP.2a] For example, if Josh is 10 and Reina is 7, how old will Reina be when Josh is 20? We cannot solve this problem with the proportion $\frac{10}{7} = \frac{20}{R}$ because it is not the case that for every 10 years that Josh ages, Reina ages 7 years. Instead, when Josh has aged 10 another years, Reina will as well, and so she will be 17 when Josh is 20.

For example, if it takes 2 people 5 hours to paint a fence, how long will it take 4 people to paint a fence of the same size (assuming all the people work at the same steady rate)? We cannot solve this problem with the proportion $\frac{2}{5} = \frac{4}{H}$ because it is not the case that for every 2 people, 5 hours of work are needed to paint the fence. When more people work, it will take fewer hours. With twice as many people working, it will take half as long, so it will take only 2.5 hours for 4 people to paint a fence. Students must understand the structure of the problem, which includes looking for and understand the roles of "for every," "for each," and "per."

Students recognize that graphs that are not lines through the origin and tables in which there is not a constant ratio in the entries do not represent proportional relationships. For example, consider circular patios that could be made with a range of diameters. For such patios, the area (and therefore the number of pavers it takes to make the patio) is not proportionally related to the diameter, although the circumference (and therefore the length of stone border it takes to encircle the patio) is proportionally related to the diameter. Note that in the case of the circumference, C, of a circle of diameter D, the constant of proportionality in $C = \pi \cdot D$ is the number π, which is not a rational number.

7.RP.2a Recognize and represent proportional relationships between quantities.

a Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

Equations for proportional relationships As students work with proportional relationships, they write equations of the form $y = cx$, where c is a constant of proportionality, i.e., a unit rate.[7.RP.2c] They

7.RP.2c Represent proportional relationships by equations.

see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table and they recognize the unit rate as the vertical increase in a "unit rate triangle" or "slope triangle" with horizontal side of length 1 for a graph of a proportional relationship.[7.RP.2b]

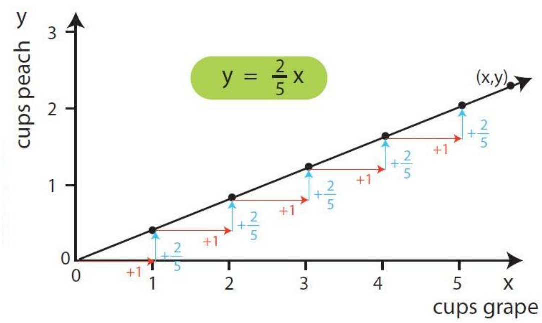
7.RP.2b Recognize and represent proportional relationships between quantities.

b Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Correspondence among a table, graph, and equation of a proportional relationship

For every 5 cups grape juice, mix in 2 cups peach juice.

x cups grape	y cups peach
(0)	(0)
5	2
1	$\frac{2}{5}$
2	$2 \cdot \frac{2}{5}$
3	$3 \cdot \frac{2}{5}$
4	$4 \cdot \frac{2}{5}$
x	$x \cdot \frac{2}{5}$



On the graph: For each 1 unit you move to the right, move up $\frac{2}{5}$ of a unit.

When you go 2 units to the right, you go up $2 \cdot \frac{2}{5}$ units.

When you go 3 units to the right, you go up $3 \cdot \frac{2}{5}$ units.

When you go 4 units to the right, you go up $4 \cdot \frac{2}{5}$ units.

When you go x units to the right, you go up $x \cdot \frac{2}{5}$ units.

Starting from $(0, 0)$, to get to a point (x, y) on the graph, go x units to the right, so go up $x \cdot \frac{2}{5}$ units.

Therefore $y = x \cdot \frac{2}{5}$ $y = \frac{2}{5}x$

Students connect their work with equations to their work with tables and diagrams. For example, if Seth runs 5 meters every 2 seconds, then how long will it take Seth to run 100 meters at that rate? The traditional method is to formulate an equation, $\frac{5}{2} = \frac{100}{T}$, cross-multiply, and solve the resulting equation to solve the problem. If $\frac{5}{2}$ and $\frac{100}{T}$ are viewed as unit rates obtained from the equivalent ratios $5 : 2$ and $100 : T$, then they must be equivalent fractions because equivalent ratios have the same unit rate. To see the rationale for cross-multiplying, note that when the fractions are given the common denominator $2 \cdot T$, then the numerators become $5 \cdot T$ and $2 \cdot 100$ respectively. Once the denominators are equal, the fractions are equal exactly when their numerators are equal, so $5 \cdot T$ must equal $2 \cdot 100$ for the unit rates to be equal. This is why we can solve the equation $5 \cdot T = 2 \cdot 100$ to find the amount of time it will take for Seth to run 100 meters.

A common error in setting up proportions is placing numbers in incorrect locations. This is especially easy to do when the order in which quantities are stated in the problem is switched within the problem statement. For example, the second of the following two problem statements is more difficult than the first because of the reversal.

> "If a factory produces 5 cans of dog food for every 3 cans of cat food, then when the company produces 600 cans of dog food, how many cans of cat food will it produce?"
>
> "If a factory produces 5 cans of dog food for every 3 cans of cat food, then how many cans of cat food will the company produce when it produces 600 cans of dog food?"

Such problems can be framed in terms of proportional relationships and the constant of proportionality or unit rate, which is obscured by the traditional method of setting up proportions. For example, if Seth runs 5 meters every 2 seconds, he runs at a rate of 2.5 meters per second, so distance d (in meters) and time t (in seconds) are related by $d = 2.5t$. If $d = 100$ then $t = \frac{100}{2.5} = 40$, so he takes 40 seconds to run 100 meters.

Multistep problems Students extend their work to solving multistep ratio and percent problems.[7.RP.3] Problems involving percent increase or percent decrease require careful attention to the referent whole. For example, consider the difference in these two percent decrease and percent increase problems:

> *Skateboard problem 1.* After a 20% discount, the price of a SuperSick skateboard is \$140. What was the price before the discount?
>
> *Skateboard problem 2.* A SuperSick skateboard costs \$140 now, but its price will go up by 20%. What will the new price be after the increase?

The solutions to these two problems are different because the 20% refers to different wholes or 100% amounts. In the first problem, the 20% is 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount. Notice that the distributive property is implicitly involved in working with percent decrease and increase. For example, in the first problem, if x is the original price of the skateboard (in dollars), then after the 20% discount, the new price is $x - 20\% \cdot x$. The distributive property shows that the new price is $80\% \cdot x$:

$$x - 20\% \cdot x = 100\% \cdot x - 20\% \cdot x = (100\% - 20\%)x = 80\% \cdot x$$

Percentages can also be used in making comparisons between two quantities. Students must attend closely to the wording of such problems to determine what the whole or 100% amount a percentage refers to.

7.RP.3 Use proportional relationships to solve multistep ratio and percent problems.

Skateboard problem 1

After a 20% discount, the price is 80% of the original price. So 80% of the original is \$140.

	percent		dollars	
÷ 4	80%	⟶	\$140	÷ 4
· 5 or add 80%+20%	20%	⟶	\$35	· 5 or add \$140+\$35
	100%	⟶	\$175	

"To find 20% I divided by 4. Then 80% plus 20% is 100%"

x = original price in dollars

	percent		dollars
discounted / original	$\frac{80}{100}$	=	$\frac{140}{x}$

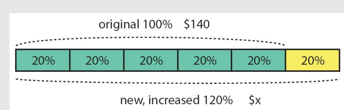
$$80x = 140 \cdot 100$$
$$x = \frac{140 \cdot 100}{80} = \frac{(2 \cdot 7 \cdot 2 \cdot 5)(2 \cdot 5 \cdot 10)}{2 \cdot 2 \cdot 2 \cdot 10} = 7 \cdot 5 \cdot 5 = 175$$

80% of the original price is \$140.

$$\frac{80}{100} \cdot x = 140$$
$$\frac{4}{5} \cdot x = 140$$
$$x = 140 \div \frac{4}{5} = 140 \cdot \frac{5}{4} = \frac{(2 \cdot 7 \cdot 2 \cdot 5) \cdot 5}{4} = 175$$

Before the discount, the price of the skateboard was \$175.

Skateboard problem 2



After a 20% increase, the price is 120% of the original price. So the new price is 120% of \$140.

	percent		dollars	
÷ 5	100%	⟶	\$140	÷ 5
· 6 or add 100%+20%	20%	⟶	\$28	· 6 or add \$140+\$28
	120%	⟶	\$168	

"To find 20% I divided by 5. Then 100% plus 20% is 120%"

x = increased price in dollars

	percent		dollars
discounted / original	$\frac{120}{100}$	=	$\frac{x}{140}$
	$\frac{12}{10}$	=	$\frac{x}{140}$

$$x = 140 \cdot \frac{12}{10} = 14 \cdot 12 = 168$$

The new, increased price is 120% of \$140.

$$x = \frac{120}{100} \cdot 140 = \frac{2 \cdot 6 \cdot 10}{2 \cdot 5 \cdot 10} \cdot 14 \cdot 2 \cdot 5 = 168$$

The new price after the increase is \$168.

Connection to Geometry One new context for proportions at Grade 7 is scale drawings.[7.G.1] To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two different figures, or they can consider the ratio of two lengths within one figure, find a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other figure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

Connection to Statistics and Probability Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.[7.SP.1] Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same characteristics. Therefore the ratio of the size of a portion having a certain characteristic to the size of the whole should be approximately the same for the sample as for the full population.

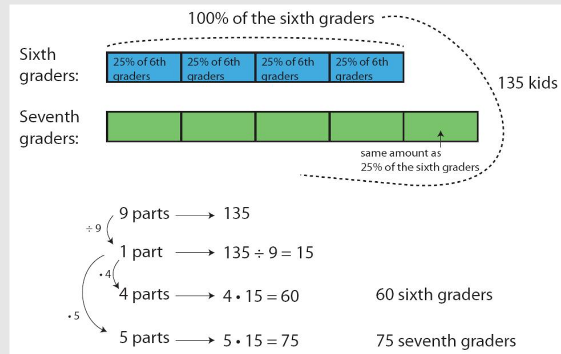
Where the Ratios and Proportional Relationships Progression is heading

The study of proportional relationships is a foundation for the study of functions, which continues through High School and beyond. Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.

Students extend their understanding of quantity. They write rates concisely in terms of derived units such as mi/hr rather than expressing them in terms such as "$\frac{3}{2}$ miles in every 1 hour." They encounter a wider variety of derived units and situations in which they must conceive units that measure attributes of interest.

Using percentages in comparisons

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?



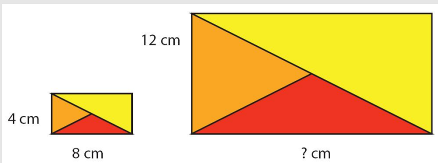
"25% more seventh graders than sixth graders means that the number of extra seventh graders is the same as 25% of the sixth graders."

[7.G.1] Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

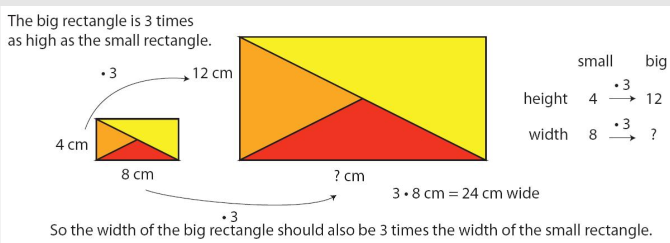
[7.SP.1] Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

Connection to geometry

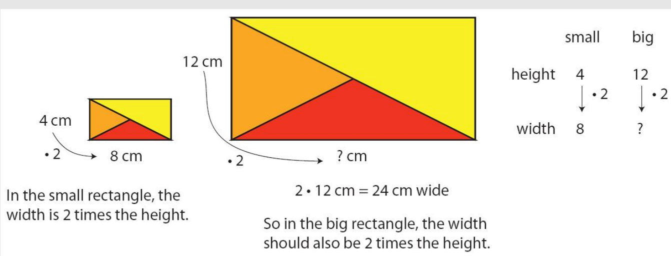
If the two rectangles are similar, then how wide is the larger rectangle?



Use a scale factor: Find the scale factor from the small rectangle to the larger one:



Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.



Connection to statistics and probability

There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I figured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2

yellow:	3	45
blue:	7	105
total:	10	150
	• 15	

"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

30% yellow tiles $\quad 30\% \cdot 150 = \frac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{3}{10} \cdot 15 \cdot 10 = 45$

70% blue tiles $\quad 70\% \cdot 150 = \frac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{7}{10} \cdot 15 \cdot 10 = 105$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."

Appendix: A framework for ratio, rate, and proportional relationships

This section presents definitions of the terms ratio, rate, and proportional relationship that are consistent with the Standards and it briefly summarizes some of the essential characteristics of these concepts. It also provides an organizing framework for these concepts. Because many different authors have used ratio and rate terminology in widely differing ways, there is a need to standardize the terminology for use with the Standards and to have a common framework for curriculum developers, professional development providers, and other education professionals to discuss the concepts. This section does not describe how the concepts should be presented to students in Grades 6 and 7.

Definitions and essential characteristics of ratios, rates, and proportional relationships

A *ratio* is a pair of non-negative numbers, $A : B$, which are not both 0.

When there are A units of one quantity for every B units of another quantity, a *rate* associated with the ratio $A : B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.) The associated *unit rate* is $\frac{A}{B}$. The term *unit rate* is the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

A rate is expressed in terms of a unit that is derived from the units of the two quantities (such as m/s, which is derived from meters and seconds). In high school and beyond, a rate is usually written as

$$\frac{A}{B}\,\frac{\text{units}}{\mathtt{\mathbf{UNITS}}}$$

where the two different fonts highlight the possibility that the quantities may have different units. In practice, when working with a ratio $A : B$, the rate $\frac{A}{B}$ units per 1 **UNIT** and the rate $\frac{B}{A}$ **UNITS** per 1 unit are both useful.

The *value* of a ratio $A : B$ is the quotient $\frac{A}{B}$ (if B is not 0). Note that the value of a ratio may be expressed as a decimal, percent, fraction, or mixed number. The value of the ratio $A : B$ tells how A and B compare multiplicatively; specifically, it tells how many times as big A is as B. In practice, when working with a ratio $A : B$, the value $\frac{A}{B}$ as well as the value $\frac{B}{A}$, associated with the ratio $B : A$, are both useful. These values of each ratio are viewed as unit rates in some contexts (see Perspective 1 in the next section).

Two ratios $A : B$ and $C : D$ are *equivalent* if there is a positive number, c, such that $C = cA$ and $D = cB$. To check that two ratios are equivalent one can check that they have the same value (because $\frac{cA}{cB} = \frac{A}{B}$), or one can "cross-multiply" and check that $A \cdot D = B \cdot C$ (because $A \cdot cB = B \cdot cA$). Equivalent ratios have the same unit rate.

A *proportional relationship* is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs (cA, cB), for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

Two perspectives on ratios and their associated rates in quantitative contexts

Although ratios, rates, and proportional relationships can be described in purely numerical terms, these concepts are most often used with quantities.

Ratios are often described as comparisons by division, especially when focusing on an associated rate or value of the ratio. There are also two broad categories of basic ratio situations. Some division situations, notably those involving area, can fit into either category of division. Many ratio situations can be viewed profitably from within either category of ratio. For this reason, the two categories for ratio will be described as *perspectives* on ratio.

First perspective: Ratio as a composed unit or batch Two quantities are in a ratio of A to B if for every A units present of the first quantity there are B units present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $c \cdot A$ units of the first quantity and $c \cdot B$ units of the second quantity. With this perspective, the two quantities can have the same or different units.

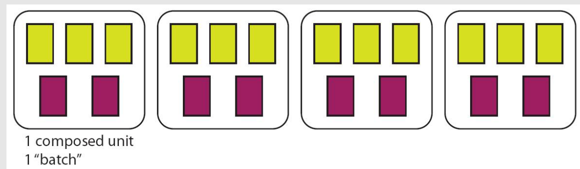
With this perspective, a ratio is specified by a composed unit or "batch," such as "3 feet in 2 seconds," and the unit or batch can be repeated or subdivided to create new pairs of amounts that are in the same ratio. For example, 12 feet in 8 seconds is in the ratio 3 to 2 because for every 3 feet, there are 2 seconds. Also, 12 feet in 8 seconds can be viewed as a 4 repetitions of the unit "3 feet in 2 seconds." Similarly, $\frac{3}{2}$ feet in 1 second is $\frac{1}{2}$ of the unit "3 feet in 2 seconds."

With this perspective, quantities that are in a ratio A to B give rise to a rate of $\frac{A}{B}$ units of the first quantity for every 1 unit of the second quantity (as well as to the rate of $\frac{B}{A}$ units of the second quantity for every 1 unit of the first quantity). For example, the ratio 3 feet in 2 seconds gives rise to the rate $\frac{3}{2}$ feet for every 1 second.

Two perspectives on ratio

1) There are 3 cups of apple juice for every 2 cups of grape juice in the mixture.

This way uses a composed unit: 3 cups apple juice and 2 cups grape juice. Any mixture that is made from some number of the composed unit is in the ratio 3 to 2.



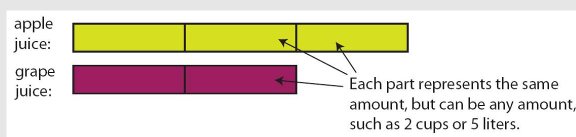
In each of these mixtures, apple juice and grape juice are mixed in a ratio of 3 to 2:

# cups apple juice	3	6	9	12	3/2	1
# cups grape juice	2	4	6	8	1	2/3

made of 2 composed units

made of 1/2 of a composed unit

2) The mixture is made from 3 parts apple juice and 2 parts grape juice, where all parts are the same size, but can be any size.



If 1 part is :	1 cup	2 cups	5 liters	3 quarts
amt of apple juice:	3 cups	6 cups	15 liters	9 quarts
amt of grape juice:	2 cups	4 cups	10 liters	6 quarts

With this perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be viewed as the numerical part of a rate associated with the ratio $A : B$.

Second perspective: Ratio as fixed numbers of parts Two quantities which have the same units, are in a ratio of A to B if there is a part of some size such that there are A parts present of the first quantity and B parts present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $A \cdot c$ units of the first quantity and $B \cdot c$ units of the second quantity.

With this perspective, one thinks of a ratio as two pieces. One piece is constituted of A parts, the other of B parts. To create pairs of measurements in the same ratio, one specifies an amount and fills each part with that amount. For example, in a ratio of 3 parts sand to 2 parts cement, each part could be filled with 5 cubic yards, so that there are 15 cubic yards of sand and 10 cubic yards of cement; or each part could be filled with 10 cubic meters, so that there are 30 cubic meters of sand and 20 cubic meters of cement. When describing a ratio from this perspective, the units need not be explicitly stated, as in "mix sand and cement in a ratio of 3 to 2." However, the type of quantity must be understood or stated explicitly, as in "by volume" or "by weight."

With this perspective, a ratio $A : B$ has an associated value, $\frac{A}{B}$, which describes how the two quantities are related multiplicatively. Specifically, $\frac{A}{B}$ is the factor that tells how many times as much of the first quantity there is as of the second quantity. (Similarly, the factor $\frac{B}{A}$ associated with the ratio $B : A$, tells how many times as much of the second quantity there is as of the first quantity.) For example, if sand and cement are mixed in a ratio of 3 to 2, then there is $\frac{3}{2}$ times as much sand as cement and there is $\frac{2}{3}$ times as much cement as sand.

With this second perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be considered a factor that does not have a unit.

Progressions for the Common Core State Standards in Mathematics (draft)

22 April 2011

6–8, Expressions and Equations

Overview

An *expression* expresses something. Facial expressions express emotions. Mathematical expressions express calculations with numbers. Some of the numbers might be given explicitly, like 2 or $\frac{3}{4}$. Other numbers in the expression might be represented by letters, such as x, y, P, or n. The calculation an expression represents might use only a single operation, as in $4+3$ or $3x$, or it might use a series of nested or parallel operations, as in $3(a+9)-9/b$. An expression can consist of just a single number, even 0.

Letters standing for numbers in an expression are called *variables*. In good practice, including in student writing, the meaning of a variable is specified by the surrounding text; an expression by itself gives no intrinsic meaning to the variables in it. Depending on the context, a variable might stand for a specific number, for example the solution to a word problem; it might be used in a universal statement true for all numbers, for example when we say that that $a+b=b+a$ for all numbers a and b; or it might stand for a range of numbers, for example when we say that $\sqrt{x^2}=x$ for $x>0$. In choosing variables to represent quantities, students specify a unit; rather than saying "let G be gasoline," they say "let G be the number of gallons of gasoline".[MP6]

[MP6] Be precise in defining variables.

An expression is a phrase in a sentence about a mathematical or real-world situation. As with a facial expression, however, you can read a lot from an algebraic expression (an expression with variables in it) without knowing the story behind it, and it is a goal of this progression for students to see expressions as objects in their own right, and to read the general appearance and fine details of algebraic expressions.

An *equation* is a statement that two expressions are equal, such as $10+0.02n=20$, or $3+x=4+x$, or $2(a+1)=2a+2$. It is an important aspect of equations that the two expressions on either side of the equal sign might not actually always be equal; that is, the equation might be a true statement for some values of the variable(s) and a false statement for others. For example,

$10 + 0.02n = 20$ is true only if $n = 500$; and $3 + x = 4 + x$ is not true for any number x; and $2(a + 1) = 2a + 2$ is true for all numbers a. A *solution* to an equation is a number that makes the equation true when substituted for the variable (or, if there is more than one variable, it is a number for each variable). An equation may have no solutions (e.g., $3 + x = 4 + x$ has no solutions because, no matter what number x is, it is not true that adding 3 to x yields the same answer as adding 4 to x). An equation may also have every number for a solution (e.g., $2(a + 1) = 2a + 2$). An equation that is true no matter what number the variable represents is called an *identity*, and the expressions on each side of the equation are said to be *equivalent expressions*. For example $2(a + 1)$ and $2a + 2$ are equivalent expressions. In Grades 6–8, students start to use properties of operations to manipulate algebraic expressions and produce different but equivalent expressions for different purposes. This work builds on their extensive experience in K–5 working with the properties of operations in the context of operations with whole numbers, decimals and fractions.

Grade 6

Apply and extend previous understandings of arithmetic to algebraic expressions Students have been writing numerical expressions since Kindergarten, such as

$$2+3 \qquad 7+6+3 \qquad 4\times(2\times 3)$$

$$8\times 5+8\times 2 \qquad \frac{1}{3}(8+7+3) \qquad \frac{3}{\frac{1}{2}}.$$

In Grade 5 they used whole number exponents to express powers of 10, and in Grade 6 they start to incorporate whole number exponents into numerical expressions, for example when they describe a square with side length 50 feet as having an area of 50^2 square feet.[6.EE.1]

6.EE.1 Write and evaluate numerical expressions involving whole-number exponents.

Students have also been using letters to represent an unknown quantity in word problems since Grade 3. In Grade 6 they begin to work systematically with algebraic expressions. They express the calculation "Subtract y from 5" as $5-y$, and write expressions for repeated numerical calculations.[MP8] For example, students might be asked to write a numerical expression for the change from a \$10 bill after buying a book at various prices:

MP8 Look for regularity in a repeated calculation and express it with a general formula.

Price of book (\$)	5.00	6.49	7.15
Change from \$10	$10-5$	$10-6.49$	$10-7.15$

Abstracting the pattern they write $10-p$ for a book costing p dollars, thus summarizing a calculation that can be carried out repeatedly with different numbers.[6.EE.2a] Such work also helps students interpret expressions. For example, if there are 3 loose apples and 2 bags of A apples each, students relate quantities in the situation to the terms in the expression $3+2A$.

6.EE.2a Write, read, and evaluate expressions in which letters stand for numbers.

a Write expressions that record operations with numbers and with letters standing for numbers.

As they start to solve word problems algebraically, students also use more complex expressions. For example, in solving the word problem

> Daniel went to visit his grandmother, who gave him \$5.50. Then he bought a book costing \$9.20. If he has \$2.30 left, how much money did he have before visiting his grandmother?

students might obtain the expression $x+5.50-9.20$ by following the story forward, and then solve the equation $x+5.50-9.20=2.30$.• Students may need explicit guidance in order to develop the strategy of working forwards, rather than working backwards from the 2.30 and calculating $2.30+9.20-5.50$.[6.EE.7] As word problems get more complex, students find greater benefit in representing the problem algebraically by choosing variables to represent quantities, rather than attempting a direct numerical solution, since the former approach provides general methods and relieves demands on working memory.

• Notice that in this problem, like many problems, a quantity, "money left," is expressed in two distinct ways:

1. starting amount + amount from grandma − amount spent
2. \$2.30

Because these two expressions refer to the same quantity in the problem situation, they are equal to each other. The equation formed by representing their equality can then be solved to find the unknown value (that is, the value of the variable that makes the equation fit the situation).

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x+p=q$ and $px=q$ for cases in which p, q and x are all nonnegative rational numbers.

Students in Grade 5 began to move from viewing expressions as actions describing a calculation to viewing them as objects in their own right;[5.OA.2] in Grade 6 this work continues and becomes more sophisticated. They describe the structure of an expression, seeing $2(8+7)$ for example as a product of two factors the second of which, $(8+7)$, can be viewed as both a single entity and a sum of two terms. They interpret the structure of an expression in terms of a context: if a runner is $7t$ miles from her starting point after t hours, what is the meaning of the 7?[MP7] If a, b, and c are the heights of three students in inches, they recognize that the coefficient $\frac{1}{3}$ in $\frac{1}{3}(a+b+c)$ has the effect of reducing the size of the sum, and they also interpret multiplying by $\frac{1}{3}$ as dividing by 3.[6.EE.2b] Both interpretations are useful in connection with understanding the expression as the mean of a, b, and c.[6.SP.3]

In the work on number and operations in Grades K–5, students have been using properties of operations to write expressions in different ways. For example, students in grades K–5 write $2+3=3+2$ and $8\times5+8\times2=8\times(5+2)$ and recognize these as instances of general properties which they can describe. They use the "any order, any grouping" property• to see the expression $7+6+3$ as $(7+3)+6$, allowing them to quickly evaluate it. The properties are powerful tools that students use to accomplish what they want when working with expressions and equations. They can be used at any time, in any order, whenever they serve a purpose.

Work with numerical expressions prepares students for work with algebraic expressions. During the transition, it can be helpful for them to solve numerical problems in which it is more efficient to hold numerical expressions unevaluated at intermediate steps. For example, the problem

> Fred and George Weasley make 150 "Deflagration Deluxe" boxes of Weasleys' Wildfire Whiz-bangs at a cost of 17 Galleons each, and sell them for 20 Galleons each. What is their profit?

is more easily solved by leaving unevaluated the total cost, 150×17 Galleons, and the total revenue 150×20 Galleons, until the subtraction step, where the distributive law can be used to calculate the answer as $150\times20-150\times17=150\times3=450$ Galleons. A later algebraic version of the problem might ask for the sale price that will yield a given profit, with the sale price represented by a letter such as p. The habit of leaving numerical expressions unevaluated prepares students for constructing the appropriate algebraic equation to solve such a problem.

As students move from numerical to algebraic work the multiplication and division symbols $\times$ and $\div$ are replaced by the conventions of algebraic notation. Students learn to use either a dot for multiplication, e.g., $1\cdot2\cdot3$ instead of $1\times2\times3$, or simple juxtaposition, e.g., $3x$ instead of $3\times x$ (during the transition, students may indicate all multiplications with a dot, writing $3\cdot x$ for $3x$). A firm grasp

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

MP7 Looking for structure in expressions by parsing them into a sequence of operations; making use of the structure to interpret the expression's meaning in terms of the quantities represented by the variables.

6.EE.2b Write, read, and evaluate expressions in which letters stand for numbers.

b Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

• The "any order, any grouping" property is a combination of the commutative and associative properties. It says that sequence of additions and subtractions may be calculated in any order, and that terms may be grouped together any way.

Some common student difficulties

- Failure to see juxtaposition as indicating multiplication, e.g., evaluating $3x$ as 35 when $x=5$, or rewriting $8-2a$ as $6a$
- Failure to see hidden 1s, rewriting $4C-C$ as 4 instead of seeing $4C-C$ as $4\cdot C-1\cdot C$ which is $3\cdot C$.

on variables as numbers helps students extend their work with the properties of operations from arithmetic to algebra.[MP2] For example, students who are accustomed to mentally calculating 5×37 as $5 \times (30 + 7) = 150 + 35$ can now see that $5(3a + 7) = 15a + 35$ for all numbers a. They apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$ and to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$.[6.EE.3]

MP2 Connect abstract symbols to their numerical referents.

6.EE.3 Apply the properties of operations to generate equivalent expressions.

Students evaluate expressions that arise from formulas used in real-world problems, such as the formulas $V = s^3$ and $A = 6s^2$ for the volume and surface area of a cube. In addition to using the properties of operations, students use conventions about the order in which arithmetic operations are performed in the absence of parentheses.[6.EE.2c] It is important to distinguish between such conventions, which are notational conveniences that allow for algebraic expressions to be written with fewer parentheses, and properties of operations, which are fundamental properties of the number system and undergird all work with expressions. In particular, the mnemonic PEMDAS• can mislead students into thinking, for example, that addition must always take precedence over subtraction because the A comes before the S, rather than the correct convention that addition and subtraction proceed from left to right (as do multiplication and division). This can lead students to make mistakes such as simplifying $n - 2 + 5$ as $n - 7$ (instead of the correct $n + 3$) because they add the 2 and the 5 before subtracting from n.[6.EE.4]

6.EE.2c Write, read, and evaluate expressions in which letters stand for numbers.

c Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

• PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, specifying the order in which operations are performed in interpreting or evaluating numerical expressions.

6.EE.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).

The order of operations tells us how to interpret expressions, but does not necessarily dictate how to calculate them. For example, the P in PEMDAS indicates that the expression $8 \times (5 + 1)$ is to be interpreted as 8 times a number which is the sum of 5 and 1. However, it does not dictate the expression must be calculated this way. A student might well see it, through an implicit use of the distributive law, as $8 \times 5 + 8 \times 1 = 40 + 8 = 48$.

The distributive law is of fundamental importance. Collecting like terms, e.g., $5b + 3b = (5 + 3)b = 8b$, should be seen as an application of the distributive law, not as a separate method.

Reason about and solve one-variable equations and inequalities In Grades K–5 students have been writing numerical equations and simple equations involving one operation with a variable. In Grade 6 they start the systematic study of equations and inequalities and methods of solving them. Solving is a process of reasoning to find the numbers which make an equation true, which can include checking if a given number is a solution.[6.EE.5] Although the process of reasoning will eventually lead to standard methods for solving equations, students should study examples where looking for structure pays off, such as in $4x + 3x = 3x + 20$, where they can see that $4x$ must be 20 to make the two sides equal.

6.EE.5 Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

This understanding can be reinforced by comparing arithmetic and algebraic solutions to simple word problems. For example, how many 44-cent stamps can you buy with \$11? Students are accustomed to solving such problems by division; now they see the parallel with representing the problem algebraically as $0.44n = 11$, from which they use the same reasoning as in the numerical solution to conclude that $n = 11 \div 0.44$.[6.EE.7]• They explore methods such as dividing both sides by the same non-zero number. As word problems grow more complex in Grades 6 and 7, analogous arithmetical and algebraic solutions show the connection between the procedures of solving equations and the reasoning behind those procedures.

When students start studying equations in one variable, it is important for them to understand every occurrence of of a given variable has the same value in the expression and throughout a solution procedure: if x is assumed to be the number satisfying the equation $4x + 3x = 3x + 20$ at the beginning of a solution procedure, it remains that number throughout.

As with all their work with variables, it is important for students to state precisely the meaning of variables they use when setting up equations (MP6). This includes specifying whether the variable refers to a specific number, or to all numbers in some range. For example, in the equation $0.44n = 11$ the variable n refers to a specific number (the number of stamps you can buy for \$11); however, if the expression $0.44n$ is presented as a general formula for calculating the price in dollars of n stamps, then n refers to all numbers in some domain.[6.EE.6] That domain might be specified by inequalities, such as $n > 0$.[6.EE.8]

Represent and analyze quantitative relationships between dependent and independent variables In addition to constructing and solving equations in one variable, students use equations in two variables to express relationships between two quantities that vary together. When they construct an expression like $10 - p$ to represent a quantity such as on page 4, students can choose a variable such as C to represent the calculated quantity and write $C = 10 - p$ to represent the relationship. This prepares students for work with functions in later grades.[6.EE.9] The variable p is the natural choice for the independent variable in this situation, with C the dependent variable. In a situation where the price, p, is to be calculated from the change, C, it might be the other way around.

As they work with such equations students begin to develop a dynamic understanding of variables, an appreciation that they can stand for any number from some domain. This use of variables arises when students study expressions such as $0.44n$, discussed earlier, or equations in two variables such as $d = 5 + 5t$ describing relationship between distance in miles, d, and time in hours, t, for a person starting 5 miles from home and walking away at 5 miles per hour. Students can use tabular• and graphical• representations to develop an appreciation of varying quantities.

6.EE.7 Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

• In Grade 7, where students learn about complex fractions, this problem can be expressed in cents as well as dollars to help students understand equivalences such as

$$\frac{11}{0.44} = \frac{1100}{44}.$$

Analogous arithmetical and algebraic solutions

J. bought three packs of balloons. He opened them and counted 12 balloons. How many balloons are in a pack?

Arithmetical solution
If three packs have twelve balloons, then one pack has $12 \div 3 = 4$ balloons.

Algebraic solution
Defining the variable: Let b be the number of balloons in a pack.
Writing the equation:

$$3b = 12$$

Solving (mirrors the reasoning of the numerical solution):

$$3b = 12 \rightarrow \frac{3b}{3} = \frac{12}{3}$$
$$b = 4.$$

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

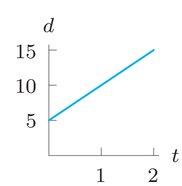
6.EE.8 Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

•

n	1	2	3	4	5
$0.44n$	0.44	0.88	1.32	1.75	2.20

•



Grade 7

Use properties of operations to generate equivalent expressions In Grade 7 students start to simplify general linear expressions• with rational coefficients. Building on work in Grade 6, where students used conventions about the order of operations to parse, and properties of operations to transform, simple expressions such as $2(3+8x)$ or $10-2p$, students now encounter linear expressions with more operations and whose transformation may require an understanding of the rules for multiplying negative numbers, such as $7-2(3-8x)$.[7.EE.1]

• A *general linear expression* in the variable x is a sum of terms which are either rational numbers, or rational numbers times x, e.g., $-\frac{1}{2}+2x+\frac{5}{8}+3x$.

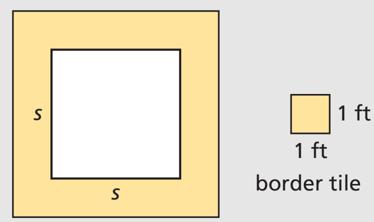
7.EE.1 Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

In simplifying this expression students might come up with answers such as

- $5(3-8x)$, mistakenly detaching the 2 from the indicated multiplication
- $7-2(-5x)$, through a determination to perform the computation in parentheses first, even though no simplification is possible
- $7-6-16x$, through an imperfect understanding of the way the distributive law works or of the rules for multiplying negative numbers.

In contrast with the simple linear expressions they see in Grade 6, the more complex expressions students seen in Grade 7 afford shifts of perspective, particularly because of their experience with negative numbers: for example, students might see $7-2(3-8x)$ as $7-(2(3-8x))$ or as $7+(-2)(3+(-8)x)$ (MP7).

Writing expressions in different forms



In expressing the number of tiles needed to border a square pool with side length s feet (where s is a whole number), students might write $4(s+1)$, $s+s+s+s+4$, or $2s+2(s+2)$, each indicating a different way of breaking up the border in order to perform the calculation. They should see all these expressions as equivalent.

As students gain experience with multiple ways of writing an expression, they also learn that different ways of writing expressions can serve different purposes and provide different ways of seeing a problem. For example, $a+0.05a=1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."[7.EE.2] In the example on the right, the connection between the expressions and the figure emphasize that they all represent the same number, and the connection between the structure of each expression and a method of calculation emphasize the fact that expressions are built up from operations on numbers.

7.EE.2 Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.

Solve real-life and mathematical problems using numerical and algebraic expressions and equations By Grade 7 students start to see whole numbers, integers, and positive and negative fractions as belonging to a single system of rational numbers, and they solve multi-step problems involving rational numbers presented in various forms.[7.EE.3]

7.EE.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. . . .

Students use mental computation and estimation to assess the reasonableness of their solutions. For example, the following statement appeared in an article about the annual migration of the Bar-tailed Godwit from Alaska to New Zealand:

> She had flown for eight days—nonstop—covering approximately 7,250 miles at an average speed of nearly 35 miles per hour.

Students can make the rough mental estimate

$$8 \times 24 \times 35 = 8 \times 12 \times 70 < 100 \times 70 = 7000$$

to recognize that although this astonishing statement is in the right ballpark, the average speed is in fact greater than 35 miles per hour, suggesting that one of the numbers in the article must be wrong.[7.EE.3]

7.EE.3 ... Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

As they build a systematic approach to solving equations in one variable, students continue to compare arithmetical and algebraic solutions to word problems. For example they solve the problem

> The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

by subtracting $2 \cdot 6$ from 54 and dividing by 2, and also by setting up the equation

$$2w + 2 \cdot 6 = 54.$$

The steps in solving the equation mirror the steps in the numerical solution. As problems get more complex, algebraic methods become more valuable. For example, in the cyclist problem in the margin, the numerical solution requires some insight in order to keep the cognitive load of the calculations in check. By contrast, choosing the letter s to stand for the unknown speed, students build an equation by adding the distances travelled in three hours ($3s$ and $3 \cdot 12.5$) and setting them equal to 63 to get

$$3s + 3 \cdot 12.5 = 63.$$

Looking for structure in word problems (MP7)

> Two cyclists are riding toward each other along a road (each at a constant speed). At 8 am, they are 63 miles apart. They meet at 11 am. If one cyclist rides at 12.5 miles per hour, what is the speed of the other cyclist?

First solution: The first cyclist travels $3 \times 12.5 = 37.5$ miles. The second travels $63 - 37.5 = 25.5$ miles, so goes $\frac{25.5}{3} = 8.5$ miles per hour. Another solution uses a key hidden quantity, the speed at which the cyclists are approaching each other, to simplify the calculations: since $\frac{63}{3} = 21$, the cyclists are approaching each other at 21 miles per hour, so the other cyclist is traveling at $21 - 12.5 = 8.5$ miles per hour.

It is worthwhile exploring two different possible next steps in the solution of this equation:

$$3s + 37.5 = 64 \quad \text{and} \quad 3(s + 12.5) = 63.$$

The first is suggested by a standard approach to solving linear equations; the second is suggested by a comparison with the numerical solution described earlier.[7.EE.4a]

7.EE.4a Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

Students also set up and solve inequalities, recognizing the ways in which the process of solving them is similar to the process of solving linear equations:

> As a salesperson, you are paid \$50 per week plus \$3 per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solution.

Students also recognize one important new consideration in solving inequalities: multiplying or dividing both sides of an inequality by a negative number reverses the order of the comparison it represents. It is useful to present contexts that allows students to make sense of this. For example,

> If the price of a ticket to a school concert is p dollars then the attendance is $1000 - 50p$. What range of prices ensures that at least 600 people attend?

Students recognize that the requirement of at least 600 people leads to the inequality $1000 - 50p \geqslant 600$. Before solving the inequality, they use common sense to anticipate that that answer will be of the form $p \leqslant ?$, since higher prices result in lower attendance.[7.EE.4b] (Note that inequalities using $\leqslant$ and $\geqslant$ are included in this standard, in addition to $>$ and $<$.)

7.EE.4b Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

b Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

Grade 8

Work with radicals and integer exponents In Grade 8 students add the properties of integer exponents to their repertoire of rules for transforming expressions.• Students have been denoting whole number powers of 10 with exponential notation since Grade 5, and they have seen the pattern in the number of zeros when powers of 10 are multiplied. They express this as $10^a 10^b = 10^{a+b}$ for whole numbers a and b. Requiring this rule to hold when a and b are integers leads to the definition of the meaning of powers with 0 and negative exponents. For example, we define $10^0 = 1$ because we want $10^a 10^0 = 10^{a+0} = 10^a$, so 10^0 must equal 1. Students extend these rules to other bases, and learn other properties of exponents.[8.EE.1]

Notice that students do not learn the properties of rational exponents until high school. However, they prepare in Grade 8 by starting to work systematically with the square root and cube root symbols, writing, for example, $\sqrt{64} = \sqrt{8^2} = 8$ and $(\sqrt[3]{5})^3 = 5$. Since $\sqrt{p}$ is defined to mean the positive solution to the equation $x^2 = p$ (when it exists), it is not correct to say (as is common) that $\sqrt{64} = \pm 8$. On the other hand, in describing the solutions to $x^2 = 64$, students can write $x = \pm\sqrt{64} = \pm 8$.[8.EE.2] Students in Grade 8 are not in a position to prove that these are the only solutions, but rather use informal methods such as guess and check.

Students gain experience with the properties of exponents by working with estimates of very large and very small quantities. For example, they estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.[8.EE.3] They express and perform calculations with very large numbers using scientific notation. For example, given that we breathe about 6 liters of air per minute, they estimate that there are $60 \times 24 = 6 \times 2.4 \times 10^2 \approx 1.5 \times 10^3$ minutes in a day, and that we therefore breath about $6 \times 1.5 \times 10^3 \approx 10^4$ liters in a day. In a lifetime of 75 years there are about $365 \times 75 \approx 3 \times 10^4$ days, and so we breath about $3 \times 10^4 \times 10^4 = 3 \times 10^8$ liters of air in a lifetime.[8.EE.4]

Understand the connections between proportional relationships, lines, and linear equations As students in Grade 8 move towards an understanding of the idea of a function, they begin to tie together a number of notions that have been developing over the last few grades:

1. An expression in one variable defines a general calculation in which the variable can represent a range of numbers—an input-output machine with the variable representing the input and the expression calculating the output. For example, $60t$ is the distance traveled in t hours by a car traveling at a constant speed of 60 miles per hour.

• **Properties of Integer Exponents**
For any nonzero rational numbers a and b and integers n and m:

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $a^n b^n = (ab)^n$
4. $a^0 = 1$
5. $a^{-n} = 1/a^n$

8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

8.EE.2 Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

8.EE.3 Use numbers expressed in the form of a single digit times a whole-number power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other.

8.EE.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

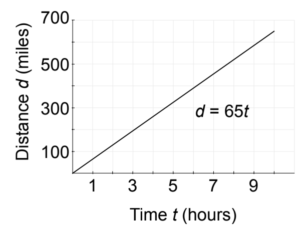
2. Choosing a variable to represent the output leads to an equation in two variables describing the relation between two quantities. For example, choosing d to represent the distance traveled by the car traveling at 65 miles per hour yields the equation $d = 65t$. Reading the expression on the right (multiplication of the variable by a constant) reveals the relationship (a rate relationship in which distance is proportional to time).

3. Tabulating values of the expression is the same as tabulating solution pairs of the corresponding equation.• This gives insight into the nature of the relationship; for example, that the distance increases by the same amount for the same increase in the time (the ratio between the two being the speed).

4. Plotting points on the coordinate plane, in which each axis is marked with a scale representing one quantity, affords a visual representation of the relationship between two quantities.•

•

t (hours)	1	2	3	4	5	6
$60t$ (miles)	60	120	180	240	300	360

• 

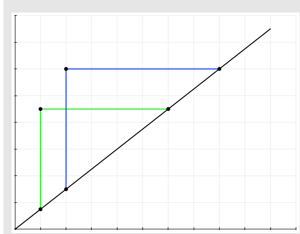
Proportional relationships provide a fruitful first ground in which these notions can grow together. The constant of proportionality is visible in each; as the multiplicative factor in the expression, as the slope of the line, and as an increment in the table (if the dependent variable goes up by 1 unit in each entry).• As students start to build a unified notion of the concept of function they are able to compare proportional relationships presented in different ways. For example, the table shows 300 miles in 5 hours, whereas the graph shows more than 300 miles in the same time.[8.EE.5]

• In the Grade 8 Functions domain, students see the relationship between the graph of a proportional relationship and its equation $y = mx$ as a special case of the relationship between a line and its equation $y = mx + b$, with $b = 0$.

[8.EE.5] Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

The connection between the unit rate in a proportional relationship and the slope of its graph depends on a connection with the geometry of similar triangles. The fact that a line has a well-defined slope—that the ratio between the rise and run for any two points on the line is always the same—depends on similar triangles.[8.EE.6]

Why lines have constant slope



The green triangle is similar to the blue triangle because corresponding angles are equal, so the ratio of rise to run is the same in each.

[8.EE.6] Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

The fact that the slope is constant between any two points on a line leads to the derivation of an equation for the line. For a line through the origin, the right triangle whose hypotenuse is the line segment from $(0, 0)$ to a point (x, y) on the line is similar to the right triangle from $(0, 0)$ to the point $(1, m)$ on the line, and so

$$\frac{y}{x} = \frac{m}{1}, \quad \text{or} \quad y = mx.$$

The equation for a line not through the origin can be derived in a similar way, starting from the y-intercept $(0, b)$ instead of the origin.

Analyze and solve linear equations and pairs of simultaneous linear equations By Grade 8 students have the tools to solve an equation which has a general linear expression on each side of the equal sign,[8.EE.7] for example:

If a bar of soap balances $\frac{3}{4}$ of a bar of soap and $\frac{3}{4}$ of a pound, how much does the bar of soap weigh?

[8.EE.7] Solve linear equations in one variable.

This is an example where choosing a letter, say b, to represent the weight of the bar of soap and solving the equation

$$b = \frac{3}{4}b + \frac{3}{4}$$

is probably easier for students than reasoning through a numerical solution. Linear equations also arise in problems where two linear functions are compared. For example

> Henry and Jose are gaining weight for football. Henry weighs 205 pounds and is gaining 2 pounds per week. Jose weighs 195 pounds and is gaining 3 pounds per week. When will they weigh the same?

Students in Grade 8 also start to solve problems that lead to simultaneous equations,[8.EE.8] for example

> Tickets for the class show are \$3 for students and \$10 for adults. The auditorium holds 450 people. The show was sold out and the class raised \$2750 in ticket sales. How many students bought tickets?

This problem involves two variables, the number S of student tickets sold and the number A of adult tickets sold, and imposes two constraints on those variables: the number of tickets sold is 450 and the dollar value of tickets sold is 2750.

8.EE.8 Analyze and solve pairs of simultaneous linear equations.

Progressions for the Common Core State Standards in Mathematics (draft)

26 December 2011

6–8 Statistics and Probability

Overview

In Grade 6, students build on the knowledge and experiences in data analysis developed in earlier grades (see K-3 Categorical Data Progression and Grades 2-5 Measurement Progression). They develop a deeper understanding of variability and more precise descriptions of data distributions, using numerical measures of center and spread, and terms such as cluster, peak, gap, symmetry, skew, and outlier. They begin to use histograms and box plots to represent and analyze data distributions. As in earlier grades, students view statistical reasoning as a four-step investigative process:

- Formulate questions that can be answered with data
- Design and use a plan to collect relevant data
- Analyze the data with appropriate methods
- Interpret results and draw valid conclusions from the data that relate to the questions posed.

Such investigations involve making sense of practical problems by turning them into statistical investigations (MP1); moving from context to abstraction and back to context (MP2); repeating the process of statistical reasoning in a variety of contexts (MP8).

In Grade 7, students move from concentrating on analysis of data to production of data, understanding that good answers to statistical questions depend upon a good plan for collecting data relevant to the questions of interest. Because statistically sound data production is based on random sampling, a probabilistic concept, students must develop some knowledge of probability before launching into sampling. Their introduction to probability is based on seeing probabilities of chance events as long-run relative frequencies of their occurrence, and many opportunities to develop the connection between theoretical probability models and empirical probability approximations. This connection forms the basis of statistical inference.

With random sampling as the key to collecting good data, students begin to differentiate between the variability in a sample and the variability inherent in a statistic computed from a sample when samples are repeatedly selected from the same population. This understanding of variability allows them to make rational decisions, say, about how different a proportion of "successes" in a sample is likely to be from the proportion of "successes" in the population or whether medians of samples from two populations provide convincing evidence that the medians of the two populations also differ.

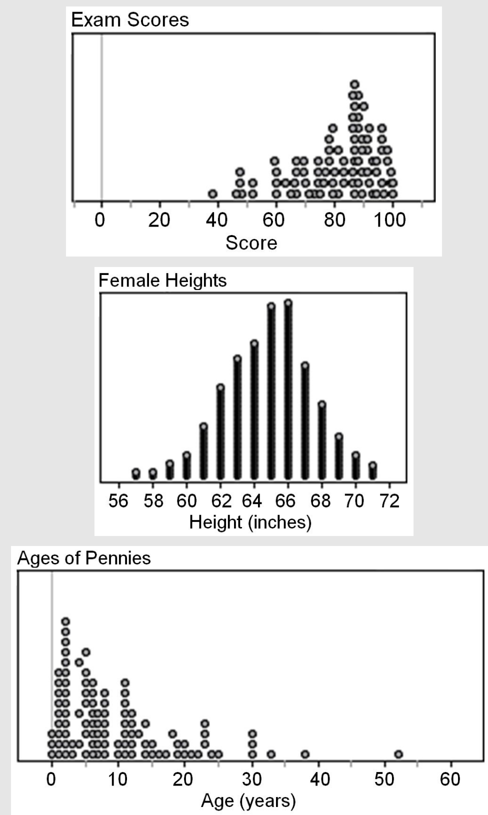
Until Grade 8, almost all of students' statistical topics and investigations have dealt with univariate data, e.g., collections of counts or measurements of one characteristic. Eighth graders apply their experience with the coordinate plane and linear functions in the study of association between two variables related to a question of interest. As in the univariate case, analysis of bivariate measurement data graphed on a scatterplot proceeds by describing shape, center, and spread. But now "shape" refers to a cloud of points on a plane, "center" refers to a line drawn through the cloud that captures the essence of its shape, and "spread" refers to how far the data points stray from this central line. Students extend their understanding of "cluster" and "outlier" from univariate data to bivariate data. They summarize bivariate categorical data using two-way tables of counts and/or proportions, and examine these for patterns of association.

Grade 6

Develop understanding of statistical variability Statistical investigations begin with a question, and students now see that answers to such questions always involve variability in the data collected to answer them.[6.SP.1] Variability may seem large, as in the selling prices of houses, or small, as in repeated measurements on the diameter of a tennis ball, but it is important to interpret variability in terms of the situation under study, the question being asked, and other aspects of the data distribution (MP2). A collection of test scores that vary only about three percentage points from 90% as compared to scores that vary ten points from 70% lead to quite different interpretations by the teacher. Test scores varying by only three points is often a good situation. But what about the same phenomenon in a different context: percentage of active ingredient in a prescription drug varying by three percentage points from order to order?

6.SP.1 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers.

Dot plots: Skewed left, symmetric, skewed right



Students distinguish between dot plots showing distributions which are skewed left (skewed toward smaller values), approximately symmetric, and skewed right (skewed toward larger values). The plots show scores on a math exam, heights of 1,000 females with ages from 18 to 24, ages of 100 pennies in a sample collected from students.

Working with counts or measurements, students display data with the dot plots (sometimes called line plots) that they used in earlier grades. New at Grade 6 is the use of histograms, which are especially appropriate for large data sets.

Students extend their knowledge of symmetric shapes,[4.G.3] to describe data displayed in dot plots and histograms in terms of symmetry. They identify clusters, peaks, and gaps, recognizing common shapes[6.SP.2] and patterns in these displays of data distributions (MP7).

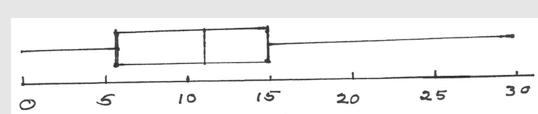
4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

6.SP.2 Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.

A major focus of Grade 6 is characterization of data distributions by measures of center and spread.[6.SP.2,6.SP.3] To be useful, center and spread must have well-defined numerical descriptions that are commonly understood by those using the results of a statistical investigation. The simpler ones to calculate and interpret are those based on counting. In that spirit, center is measured by the *median*, a number arrived at by counting to the middle of an ordered array of numerical data. When the number of data points is odd, the median is the middle value. When the number of data points is even, the median is the average of the two middle values. *Quartiles*, the medians of the lower and upper halves of the ordered data values, mark off the middle 50% of the data values and, thus, provide information on the spread of the data.[1] The distance between the first and third quartiles, the *interquartile range* (IQR), is a single number summary that serves as a very useful measure of variability.[6.SP.3]

Plotting the extreme values, the quartiles, and the median (the *five-number summary*) on a number line diagram, leads to the *box plot*, a concise way of representing the main features of a data dis-

Box plot



For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 30}, the median is 11 (from the average of the two middle values 10 and 12), the interquartile range is $15 - 6 = 9$, and the extreme values are 1 and 30.

[1] Different methods for computing quartiles are in use. The Standards uses the method which excludes the median to create two halves when the number of data points is odd. See Langford, "Quartiles in Elementary Statistics," *Journal of Statistics Education*, 2006, for a description of the different methods used by statisticians and statistical software.

tribution.• Box plots are particularly well suited for comparing two or more data sets, such as the lengths of mung bean sprouts for plants with no direct sunlight versus the lengths for plants with four hours of direct sunlight per day.[6.SP.4]

Students use their knowledge[6.NS.2,6.NS.3] of division, fractions, and decimals in computing a new measure of center—the *arithmetic mean*, often simply called the *mean*. They see the mean as a "leveling out" of the data in the sense of a unit rate (see Ratio and Proportion Progression). In this "leveling out" interpretation, the mean is often called the "average" and can be considered in terms of "fair share." For example, if it costs a total of $40 for five students to go to lunch together and they decide to pay equal shares of the cost, then each student's share is $8.00. Students recognize the mean as a convenient summary statistic that is used extensively in the world around them, such as average score on an exam, mean temperature for the day, average height and weight of a person of their age, and so on.

Students also learn some of the subtleties of working with the mean, such as its sensitivity to changes in data values and its tendency to be pulled toward an extreme value, much more so than the median. Students gain experience in deciding whether the mean or the median is the better measure of center in the context of the question posed. Which measure will tend to be closer to where the data on prices of a new pair of jeans actually cluster? Why does your teacher report the mean score on the last exam? Why does your science teacher say, "Take three measurements and report the average?"

For distributions in which the mean is the better measure of center, variation is commonly measured in terms of how far the data values deviate from the mean. Students calculate how far each value is above or below the mean, and these deviations from the mean are the first step in building a measure of variation based on spread to either side of center. The average of the deviations is always zero, but averaging the absolute values of the deviations leads to a measure of variation that is useful in characterizing the spread of a data distribution and in comparing distributions. This measure is called the *mean absolute deviation*, or MAD. Exploring variation with the MAD sets the stage for introducing the standard deviation in high school.

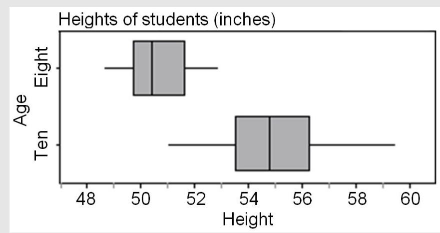
Summarize and describe distributions "How many text messages do middle school students send in a typical day?" Data obtained from a sample of students may have a distribution with a few very large values, showing a "long tail" in the direction of the larger values. Students realize that the mean may not represent the largest cluster of data points, and that the median is a more useful measure of center. In like fashion, the IQR is a more useful measure of spread, giving the spread of the middle 50% of the data points.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

• "Box plot" is also sometimes written "boxplot." Because of the different methods for computing quartiles and other different conventions, there are different kinds of box plots in use. Box plots created from the five-number summary do not show points detached from the remainder of the diagram. However, box plots generated with statistical software may display these features.

6.SP.4 Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

Comparing distributions with box plots

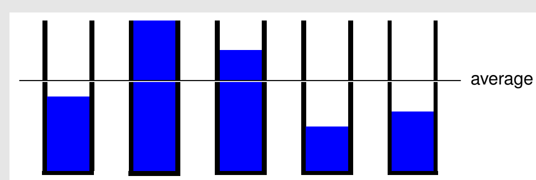


In Grade 6, box plots can be used to analyze the data from Example 2 of the Measurement Data Progression. Sixth graders can give more precise answers in terms of center and spread to questions asked at earlier grades. "Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?"

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

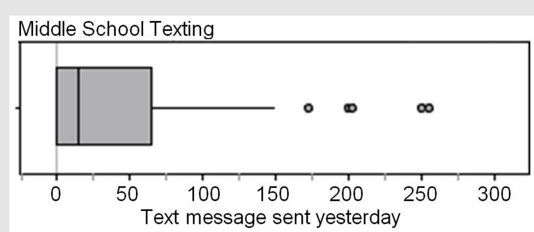
6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Average as a "leveling out"



As mentioned in the Grades 2-5 Measurement Data Progression, students in Grade 5 might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. In Grade 6, students are able to view the amount in each cylinder after redistribution as equal to the mean of the five original amounts.

Middle School Texting

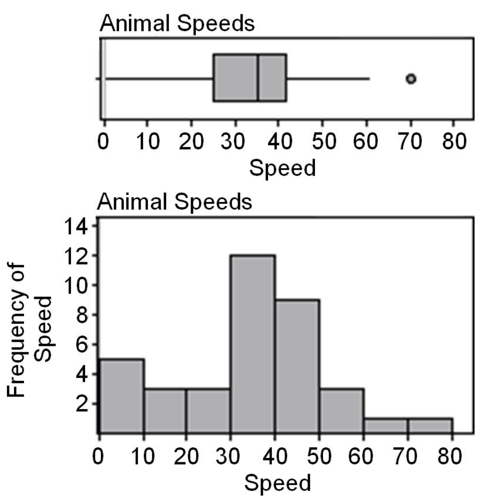


The 37 animal speeds shown in the margin can be used to illustrate summarizing a distribution.[6.SP.5a–c] According to the source, "Most of the following measurements are for maximum speeds over approximate quarter-mile distances. Exceptions—which are included to give a wide range of animals—are the lion and elephant, whose speeds were clocked in the act of charging; the whippet, which was timed over a 200-yard course; the cheetah over a 100-yard distance; humans for a 15-yard segment of a 100-yard run; and the black mamba snake, six-lined race runner, spider, giant tortoise, three-toed sloth, . . . , which were measured over various small distances." Understanding that it is difficult to measure speeds of wild animals, does this description raise any questions about whether or not this is a fair comparison of the speeds?

Moving ahead with the analysis, students will notice that the distribution is not symmetric, but the lack of symmetry is mild. It is most appropriate to measure center with the median of 35 mph and spread with the IQR of $42 - 25 = 17$. That makes the cheetah an outlier with respect to speed, but notice again the description of how this speed was measured. If the garden snail with a speed of 0.03 mph is added to the data set, then cheetah is no longer considered an outlier. Why is that?

Because the lack of symmetry is not severe, the mean (32.15 mph) is close to the median and the MAD (12.56 mph) is a reasonable measure of typical variation from the mean, as about 57% of the data values lie within one MAD of the mean, an interval from about 19.6 mph to 44.7 mph.

Box plot and histogram of 37 animal speeds



Note that the isolated point (the extreme value of 70 mph) has been generated by the software used to produce the box plot. The mild lack of symmetry can be seen in the box plot in the median (slightly off-center in the box) and in the slightly different lengths of the "whiskers." The geometric shape made by the histogram also shows mild lack of symmetry.

6.SP.5a Summarize numerical data sets in relation to their context, such as by:

a Reporting the number of observations.

6.SP.5b Summarize numerical data sets in relation to their context, such as by:

b Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

6.SP.5c Summarize numerical data sets in relation to their context, such as by:

c Giving quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

Table of 37 animal speeds

Animal	Speed (mph)
Cheetah	70.00
Pronghorn antelope	61.00
Lion	50.00
Thomson's gazelle	50.00
Wildebeest	50.00
Quarter horse	47.50
Cape hunting dog	45.00
Elk	45.00
Coyote	43.00
Gray fox	42.00
Hyena	40.00
Ostrich	40.00
Zebra	40.00
Mongolian wild ass	40.00
Greyhound	39.35
Whippet	35.50
Jackal	35.00
Mule deer	35.00
Rabbit (domestic)	35.00
Giraffe	32.00
Reindeer	32.00
Cat (domestic)	30.00
Kangaroo	30.00
Grizzly bear	30.00
Wart hog	30.00
White-tailed deer	30.00
Human	27.89
Elephant	25.00
Black mamba snake	20.00
Six-lined race runner	18.00
Squirrel	12.00
Pig (domestic)	11.00
Chicken	9.00
House mouse	8.00
Spider (Tegenearia atrica)	1.17
Giant tortoise	0.17
Three-toed sloth	0.15

Source: `factmonster.com/ipka/A0004737.html`

Grade 7

Chance processes and probability models In Grade 7, students build their understanding of probability on a relative frequency view of the subject, examining the proportion of "successes" in a chance process—one involving repeated observations of random outcomes of a given event, such as a series of coin tosses. "What is my chance of getting the correct answer to the next multiple choice question?" is not a probability question in the relative frequency sense. "What is my chance of getting the correct answer to the next multiple choice question *if I make a random guess among the four choices?*" is a probability question because the student could set up an experiment of multiple trials to approximate the relative frequency of the outcome.• And two students doing the same experiment will get nearly the same approximation. These important points are often overlooked in discussions of probability.[7.SP.5]

• Note the connection with MP6. Including the stipulation "if I make a random guess among the four choices" makes the question precise enough to be answered with the methods discussed for this grade.

7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

Students begin by relating probability to the long-run (more than five or ten trials) relative frequency of a chance event, using coins, number cubes, cards, spinners, bead bags, and so on. Hands-on activities with students collecting the data on probability experiments are critically important, but once the connection between observed relative frequency and theoretical probability is clear, they can move to simulating probability experiments via technology (graphing calculators or computers).

It must be understood that the connection between relative frequency and probability goes two ways. If you know the structure of the generating mechanism (e.g., a bag with known numbers of red and white chips), you can anticipate the relative frequencies of a series of random selections (with replacement) from the bag. If you do not know the structure (e.g., the bag has unknown numbers of red and white chips), you can approximate it by making a series of random selections and recording the relative frequencies.[7.SP.6] This simple idea, obvious to the experienced, is essential and not obvious at all to the novice.• The first type of situation, in which the structure is known, leads to "probability"; the second, in which the structure is unknown, leads to "statistics."

7.SP.6 Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability.

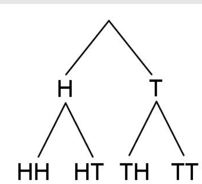
• Examples of student strategies for generalizing from the relative frequency in the simplest case (one sample) to the relative frequency in the whole population are given in the Ratio and Proportional Relationship Progression, p. 11.

A *probability model* provides a probability for each possible non-overlapping outcome for a chance process so that the total probability over all such outcomes is unity. The collection of all possible individual outcomes is known as the *sample space* for the model. For example, the sample space for the toss of two coins (fair or not) is often written as {TT, HT, TH, HH}. The probabilities of the model can be either *theoretical* (based on the structure of the process and its outcomes) or *empirical* (based on observed data generated by the process). In the toss of two balanced coins, the four outcomes of the sample space are given equal theoretical probabilities of $\frac{1}{4}$ because of the symmetry of the process—because the coins are balanced, an outcome of heads is just as likely as an outcome of tails. Randomly selecting a name from a list of ten students also leads to equally

Different representations of a sample space

HH
HT
TH
TT

	H	T
H	HH	HT
T	TH	TT



All the possible outcomes of the toss of two coins can be represented as an organized list, table, or tree diagram. The sample space becomes a probability model when a probability for each simple event is specified.

likely outcomes with probability 0.10 that a given student's name will be selected.[7.SP.7a] If there are exactly four seventh graders on the list, the chance of selecting a seventh grader's name is 0.40. On the other hand, the probability of a tossed thumbtack landing point up is not necessarily $\frac{1}{2}$ just because there are two possible outcomes; these outcomes may not be equally likely and an empirical answer must be found be tossing the tack and collecting data.[7.SP.7b]

The product rule for counting outcomes for chance events should be used in finite situations like tossing two or three coins or rolling two number cubes. There is no need to go to more formal rules for permutations and combinations at this level. Students should gain experience in the use of diagrams, especially trees and tables, as the basis for organized counting of possible outcomes from chance processes.[7.SP.8] For example, the 36 equally likely (theoretical probability) outcomes from the toss of a pair of number cubes are most easily listed on a two-way table. An archived table of census data can be used to approximate the (empirical) probability that a randomly selected Florida resident will be Hispanic.

After the basics of probability are understood, students should experience setting up a model and using simulation (by hand or with technology) to collect data and estimate probabilities for a real situation that is sufficiently complex that the theoretical probabilities are not obvious. For example, suppose, over many years of records, a river generates a spring flood about 40% of the time. Based on these records, what is the chance that it will flood for at least three years in a row sometime during the next five years?[7.SP.8c]

Random sampling In earlier grades students have been using data, both categorical and measurement, to answer simple statistical questions, but have paid little attention to how the data were selected. A primary focus for Grade 7 is the process of selecting a random sample, and the value of doing so. If three students are to be selected from the class for a special project, students recognize that a fair way to make the selection is to put all the student names in a box, mix them up, and draw out three names "at random." Individual students realize that they may not get selected, but that each student has the same chance of being selected. In other words, random sampling is a fair way to select a subset (a sample) of the set of interest (the population). A statistic computed from a random sample, such as the mean of the sample, can be used as an estimate of that same characteristic of the population from which the sample was selected. This estimate must be viewed with some degree of caution because of the variability in both the population and sample data. A basic tenet of statistical reasoning, then, is that random sampling allows results from a sample to be generalized to a much larger body of data, namely, the population from which the sample was selected.[7.SP.1]

"What proportion of students in the seventh grade of your school

7.SP.7a Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

a Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events.

7.SP.7b Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.

b Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process.

7.SP.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

7.SP.8c Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

c Design and use a simulation to generate frequencies for compound events.

7.SP.1 Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

choose football as their favorite sport?" Students realize that they do not have the time and energy to interview all seventh graders, so the next best way to get an answer is to select a random sample of seventh graders and interview them on this issue. The sample proportion is the best estimate of the population proportion, but students realize that the two are not the same and a different sample will give a slightly different estimate. In short, students realize that conclusions drawn from random samples generalize beyond the sample to the population from which the sample was selected, but a sample *statistic* is only an estimate of a corresponding population *parameter* and there will be some discrepancy between the two. Understanding variability in sampling allows the investigator to gauge the expected size of that discrepancy.

The variability in samples can be studied by means of simulation.[7.SP.2] Students are to take a random sample of 50 seventh graders from a large population of seventh graders to estimate the proportion having football as their favorite sport. Suppose, for the moment, that the true proportion is 60%, or 0.60. How much variation can be expected among the sample proportions? The scenario of selecting samples from this population can be simulated by constructing a "population" that has 60% red chips and 40% blue chips, taking a sample of 50 chips from that population, recording the number of red chips, replacing the sample in the population, and repeating the sampling process. (This can be done by hand or with the aid of technology, or by a combination of the two.) Record the proportion of red chips in each sample and plot the results.

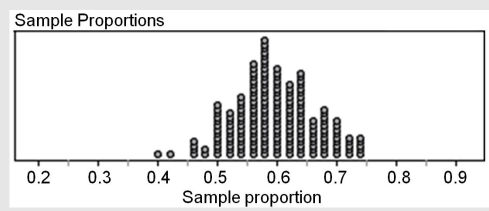
The dot plots in the margin shows results for 200 such random samples of size 50 each. Note that the sample proportions pile up around 0.60, but it is not too rare to see a sample proportion down around 0.45 or up around .0.75. Thus, we might expect a variation of close to 15 percentage points in either direction. Interestingly, about that same amount of variation persists for true proportions of 50% and 40%, as shown in the dot plots.

Students can now reason that random samples of size 50 are likely to produce sample proportions that are within about 15 percentage points of the true population value. They should now conjecture as to what will happen of the sample size is doubled or halved, and then check out the conjectures with further simulations. Why are sample sizes in public opinion polls generally around 1000 or more, rather than as small as 50?

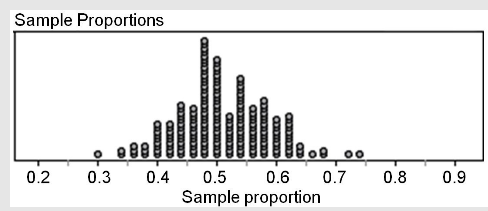
Informal comparative inference To estimate a population mean or median, the best practice is to select a random sample from that population and use the sample mean or median as the estimate, just as with proportions. But, many of the practical problems dealing with measures of center are comparative in nature, as in comparing average scores on the first and second exam or comparing average salaries between female and male employees of a firm. Such

7.SP.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions.

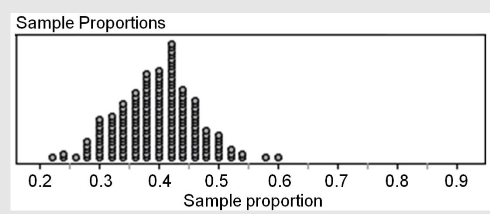
Results of simulations



Proportions of red chips in 200 random samples of size 50 from a population in which 60% of the chips are red.



Proportions of red chips in 200 random samples of size 50 from a population in which 50% of the chips are red.



Proportions of red chips in 200 random samples of size 50 from a population in which 40% of the chips are red.

comparisons may involve making conjectures about population parameters and constructing arguments based on data to support the conjectures (MP3).

If all measurements in a population are known, no sampling is necessary and data comparisons involve the calculated measures of center. Even then, students should consider variability.[7.SP.3] The figures in the margin show the female life expectancies for countries of Africa and Europe. It is clear that Europe tends to have the higher life expectancies and a much higher median, but some African countries are comparable to some of those in Europe. The mean and MAD for Africa are 53.6 and 9.5 years, respectively, whereas those for Europe are 79.3 and 2.8 years. In Africa, it would not be rare to see a country in which female life expectancy is about ten years away from the mean for the continent, but in Europe the life expectancy in most countries is within three years of the mean.

For random samples, students should understand that medians and means computed from samples will vary from sample to sample and that making informed decisions based on such sample statistics requires some knowledge of the amount of variation to expect. Just as for proportions, a good way to gain this knowledge is through simulation, beginning with a population of known structure.

The following examples are based on data compiled from nearly 200 middle school students in the Washington, DC area participating in the Census at Schools Project. Responses to the question, "How many hours per week do you usually spend on homework?," from a random sample of 10 female students and another of 10 male students from this population gave the results plotted in the margin.

Females have a slightly higher median, but students should realize that there is too much variation in the sample data to conclude that, in this population, females have a higher median homework time. An idea of how much variation to expect in samples of size 10 is needed.

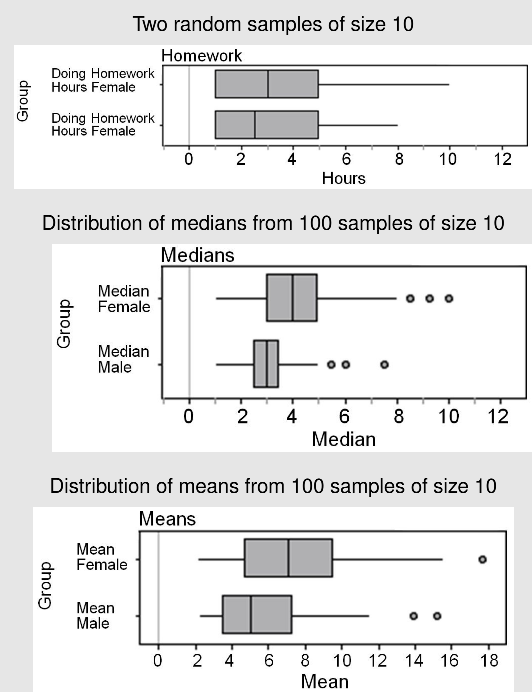
Simulation to the rescue! Students can take multiple samples of size 10 from the Census of Schools data to see how much the sample *medians* themselves tend to vary.[7.SP.4] The sample medians for 100 random samples of size 10 each, with 100 samples of males and 100 samples of females, is shown in the margin. This plot shows that the sample medians vary much less than the homework hours themselves and provides more convincing evidence that the female median homework hours is larger than that for males. Half of the female sample medians are within one hour of 4 while half of the male sample medians are within half hour of 3, although there is still overlap between the two groups.

A similar analysis based on sample means gave the results seen in the margin. Here, the overlap of the two distributions is more severe and the evidence weaker for declaring that the females have higher mean study hours than males.

[7.SP.3] Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability.

Female life expectancies in African and European countries

Hours spent on homework per week



Source: Census at Schools Project, `amstat.org/censusatschool/`

[7.SP.4] Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations.

Grade 8

Investigating patterns of association in bivariate data Students now have enough experience with coordinate geometry and linear functions[8.F.3,8.F.4,8.F.5] to plot bivariate data as points on a plane and to make use of the equation of a line in analyzing the relationship between two paired variables. They build statistical models to explore the relationship between two variables (MP4); looking for and making use of structure to describe possible association in bivariate data (MP7).

Working with paired measurement variables that might be associated linearly or in a more subtle fashion, students construct a scatter plot, describing the pattern in terms of clusters, gaps, and unusual data points (much as in the univariate situation). Then, they look for an overall positive or negative trend in the cloud of points, a linear or nonlinear (curved) pattern, and strong or weak association between the two variables, using these terms in describing the nature of the observed association between the variables.[8.SP.1]

For a data showing a linear pattern, students sketch a line through the "center" of the cloud of points that captures the essential nature of the trend, at first by use of an informal fitting procedure, perhaps as informal as laying a stick of spaghetti on the plot. How well the line "fits" the cloud of points is judged by how closely the points are packed around the line, considering that one or more outliers might have tremendous influence on the positioning of the line.[8.SP.2]

After a line is fit through the data, the slope of the line is approximated and interpreted as a rate of change, in the context of the problem.[8.F.4] The slope has important practical interpretations for most statistical investigations of this type (MP2). On the Exam 1 versus Exam 2 plot, what does the slope of 0.6 tell you about the relationship between these two sets of scores? Which students tend to do better on the second exam and which tend to do worse?[8.SP.3] Note that the negative linear trend in mammal life spans versus speed is due entirely to three long-lived, slow animals (hippo, elephant, and grizzly bear) and one short-lived, fast one (cheetah). Students with good geometry skills might explain why it would be unreasonable to expect that alligator lengths and weights would be linearly related.

Building on experience with decimals and percent, and the ideas of association between measurement variables, students now take a more careful look at possible association between categorical variables.[8.SP.4] "Is there a difference between sixth graders and eighth graders with regard to their preference for rock, rap, or country music?" Data from a random sample of sixth graders and another random sample of eighth graders are summarized by frequency counts in each cell in a two-way table of preferred music type by grade. The proportions of favored music type for the sixth graders are then compared to the proportions for eighth graders. If the two proportions for each music type are about the same, there is little or no

8.F.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

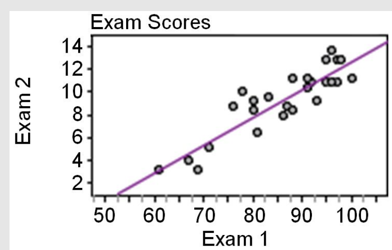
8.F.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

8.F.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

8.SP.1 Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

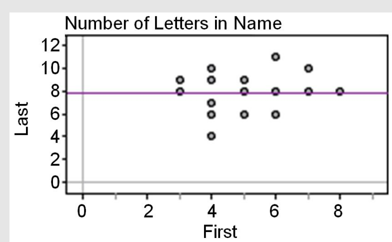
8.SP.2 Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

Scores on Exam 1 and Exam 2



The least squares line fitted to the points has a positive slope and the points are closely clustered about the line, thus, the scores said to show strong positive association. Students with high scores on one exam tend to have high scores on the other. Students with low scores on one exam tend to have low scores on the other.

Letters in first and last names of students



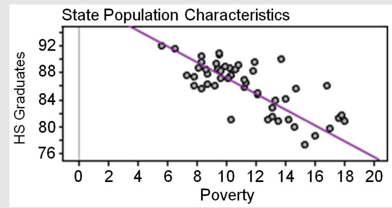
The line fitted to the points is horizontal. The number of letters in a student's first name shows no association with the number of letters in a student's last name.

8.SP.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

association between the grade and music preference because both grades have about the same preferences. If the two proportions differ, there is some evidence of association because grade level seems to make a difference in music preferences. The nature of the association should then be described in more detail.

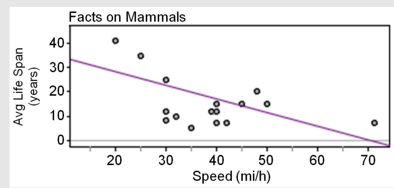
The table in the margin shows percentages of U.S. residents who have health risks due to obesity, by age category. Students should be able to explain what the cell percentages represent and provide a clear description of the nature of the association between the variables *obesity risk* and *age*. Can you tell, from this table alone, what percentage of those over the age of 18 are at risk from obesity? Such questions provide a practical mechanism for reinforcing the need for clear understanding of proportions and percentages.

High school graduation and poverty percentages for states



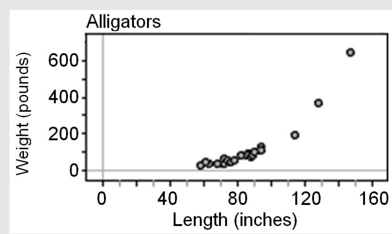
The line fitted to the data has a negative slope and data points are not all tightly clustered about the line. The percentage of a state's population in poverty shows a moderate negative association with the percentage of a state's high school graduates.

Average life span and speeds of mammals



The negative trend is due to a few outliers. This as can be seen by examining the effect of removing those points.

Weight versus length of Florida alligators



Source: http://www.factmonster.com/ipka/A0004737.html

A nonlinear association.

Table schemes for comparing frequencies and row proportions

	Rock	Rap	Country	Total
6th graders	a	b	c	d
8th graders	e	f	g	h

	Rock	Rap	Country	Total
6th graders	a/d	b/d	c/d	d
8th graders	e/h	f/h	g/h	h

Each letter represents a frequency count.

Obesity risk percentages

Age Category	Obesity		
	Not At Risk	At Risk	Row Total
Age 18 to 24	57.3	42.7	100
Age 25 to 44	38.6	61.4	100

Source: Behavioral Risk Factor Surveillance System of the Center for Disease Control

Where the Statistics and Probability Progression is heading

In high school, students build on their experience from the middle grades with data exploration and summarization, randomization as the basis of statistical inference, and simulation as a tool to understand statistical methods.

Just as Grade 6 students deepen the understanding of univariate data initially developed in elementary school, high school students deepen their understanding of bivariate data, initially developed in middle school. Strong and weak association is expressed more precisely in terms of correlation coefficients, and students become familiar with an expanded array of functions in high school that they use in modeling association between two variables.

They gain further familiarity with probability distributions generated by theory or data, and use these distributions to build an empirical understanding of the normal distribution, which is the main distribution used in measuring sampling error. For statistical methods related to the normal distribution, variation from the mean is measured by standard deviation.

Students extend their knowledge of probability, learning about conditional probability, and using probability distributions to solve problems involving expected value.

Progressions for the Common Core State Standards in Mathematics (draft)

4 July 2013

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For updates and more information about the Progressions, see http://ime.math.arizona.edu/progressions.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Number System, 6–8

Overview

In Grades 6–8, students build on two important conceptions which have developed throughout K–5, in order to understand the rational numbers as a number system. The first is the representation of whole numbers and fractions as points on the number line, and the second is a firm understanding of the properties of operations on whole numbers and fractions.

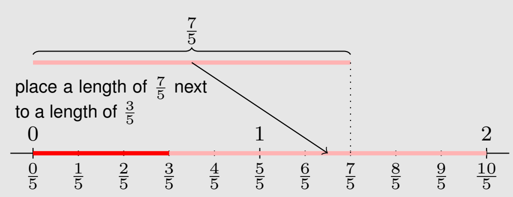
Representing numbers on the number line In early grades, students see whole numbers as counting numbers, Later, they also understand whole numbers as corresponding to points on the number line. Just as the 6 on a ruler measures 6 inches from the 0 mark, so the number 6 on the number line measures 6 units from the origin. Interpreting numbers as points on the number line brings fractions into the family as well; fractions are seen as measurements with new units, creating by partitioning the whole number unit into equal pieces. Just as on a ruler we might measure in tenths of an inch, on the number line we have halves, thirds, fifths, sevenths; the number line is a sort of ruler with every denominator. The denominators 10, 100, etc. play a special role, partioning the number line into tenths, hundredths, etc., just as a metric ruler is partioned into centimeters and millimeters.

Starting in Grade 2 students see addition as concatenation of lengths on the number line.[2.MD.6] By Grade 4 they are using the same model to represent the sum of fractions with the same denominator: $\frac{3}{5}+\frac{7}{5}$ is seen as putting together a length that is 3 units of one fifth long with a length that is 7 units of one fifth long, making 10 units of one fifths in all. Since there are five fifths in 1 (that's what it means to be a fifth), and 10 is 2 fives, we get $\frac{3}{5}+\frac{7}{5}=2$. Two fractions with different denominators are added by representing them in terms of a common unit.

Representing sums as concatenated lengths on the number line is important because it gives students a way to think about addition that makes sense independently of how numbers are represented symbolically. Although addition calculations may look different for numbers represented in base ten and as fractions, addition is the

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Representing $\frac{3}{5}$ and $\frac{7}{5}$ on the number line



same operation in each case. Furthermore, the concatenation model of addition extends naturally to negative numbers in Grade 7.

Properties of operations The number line provides a representation that can be used to building understanding of sums and differences of rational numbers. However, building understanding of multiplication and division of rational numbers relies on a firm understanding of properties of operations. Although students have not necessarily been taught formal names for these properties, they have used them repeatedly in elementary school and have been with reasoning with them. The commutative and associative properties of addition and mutiplication have, in particular, been their constant friends in working with strategies for addition and multiplication.[1.OA.3, 3.OA.5]

The existence of the multiplicative identity (1) and multiplicative inverses start to play important roles as students learn about fractions. They might see fraction equivalence as confirming the identity rule for fractions. In Grade 4 they learn about fraction equivalence

$$\frac{n \times a}{n \times b} = \frac{a}{b}$$

and in Grade 5 they relate this to multiplication by 1

$$\frac{n \times a}{n \times b} = \frac{n}{n} \times \frac{a}{b} = 1 \times \frac{a}{b},$$

thus confirming that the identity rule

$$1 \times \frac{a}{b} = \frac{a}{b}$$

works for fractions.[5.NF.5]

As another example, the commutative property for multiplication plays an important role in understanding multiplication with fractions. For example, although

$$5 \times \frac{1}{2} = \frac{5}{2}$$

can be made sense of using previous understandings of whole number multiplication as repeated addition, the other way around,

$$\frac{1}{2} \times 5 = \frac{5}{2},$$

seems to come from a different source, from the meaning of phrases such as "half of" and a mysterious acceptance that "of" must mean multiplication. A more reasoned approach would be to observe that if we want the commutative property to continue to hold, then we must have

$$\frac{1}{2} \times 5 = 5 \times \frac{1}{2} = \frac{5}{2},$$

Properties of Operations on Rational Numbers

Properties of Addition

1. Commutative Property. For any two rational numbers a and b, $a + b = b + a$.
2. Associative Property. For any three rational numbers a, b and c, $(a + b) + c = a + (b + c)$.
3. Existence of Identity. The number 0 satisfies $0 + a = a = a + 0$.
4. Existence of Additive Inverse. For any rational number a, there is a number $-a$ such that $a + (-a) = 0$.

Properties of Multiplication

1. Commutative Property. For any two rational numbers a and b, $a \times b = b \times a$.
2. Associative Property. For any three rational numbers a, b and c, $(a \times b) \times c = a \times (b \times c)$.
3. Existence of Identity. The number 1 satisfies $1 \times a = a = a \times 1$.
4. Existence of Multiplicative Inverse. For every non-zero rational number a, there is a rational number $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$.

The Distributive Property

For rational numbers a, b and c, one has $a \times (b + c) = a \times b + a \times c$.

1.OA.3 Apply properties of operations as strategies to add and subtract.[1]

3.OA.5 Apply properties of operations as strategies to multiply and divide.[2]

5.NF.5 Interpret multiplication as scaling (resizing), by:

a ...

b ...and relating the principle of fraction equivalence $\frac{a}{b} = \frac{n \times a}{n \times b}$ to the effect of multiplying $\frac{a}{b}$ by 1.

and that $\frac{5}{2}$ is indeed "half of five," as we have understood in Grade 5.[5.NF.3]

When students extend their conception of multiplication to include negative rational numbers, the properties of operations become crucial. The rule that the product of negative numbers is positive, often seen as mysterious, is the result of extending the properties of operations (particularly the distributive property) to rational numbers.

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

Grade 6

As Grade 6 dawns, students have a firm understanding of place value and the properties of operations. On this foundation they are ready to start using the properties of operations as tools of exploration, deploying them confidently to build new understandings of operations with fractions and negative numbers. They are also ready to complete their growing fluency with algorithms for the four operations.

Apply and extend previous understandings of multiplication and division to divide fractions by fractions In Grade 6 students conclude the work with operations on fractions, started in Grade 4, by computing quotients of fractions.[6.NS.1] In Grade 5 students divided unit fractions by whole numbers and whole numbers by unit fractions, two special cases of fraction division that are relatively easy to conceptualize and visualize.[5.NF.7ab] Dividing a whole number by a unit fraction can be conceptualized in terms of the measurement interpretation of division, which conceptualizes $a \div b$ as the the measure of of a by units of length b on the number line, that is, the solution to the multiplication equation $a = ? \times b$. Dividing a unit fraction by a whole number can be interpreted in terms of the sharing interpretation of division, which conceptualizes $a \div b$ as the size of a share when a is divided into b equal shares, that is, the solution to the multiplication equation $a = b \times ?$.

Now in Grade 6 students develop a general understanding of fraction division. They can use story contexts and visual models to develop this understanding, but also begin to move towards using the relation between division and multiplication.

For example, they might use the measurement interpretation of division to see that $\frac{8}{3} \div \frac{2}{3} = 4$, because 4 is 4 is how many $\frac{2}{3}$ there are in $\frac{8}{3}$. At the same time they can see that this latter statement also says that $4 \times \frac{2}{3} = \frac{8}{3}$. This multiplication equation can be used to obtain the division equation directly, using the relation between multiplication and division.

Quotients of fractions that are whole number answers are particularly suited to the measurement interpretation of division. When this interpretation is used for quotients of fractions that are not whole numbers, it can be rephrased from "how many times does this go into that?" to "how much of this is in that?" For example,

$$\frac{2}{3} \div \frac{3}{4}$$

can be interpreted as how many $\frac{3}{4}$-cup servings are in $\frac{2}{3}$ of a cup of yogurt, or as how much of a $\frac{3}{4}$-cup serving is in $\frac{2}{3}$ of a cup of yogurt. Although linguistically different the two questions are mathematically the same. Both can be visualized as in the margin and expressed using a multiplication equation with an unknown for the

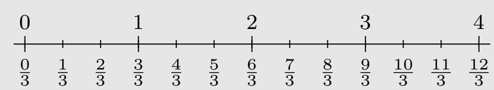
6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

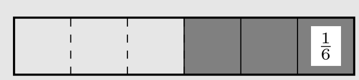
a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

b Interpret division of a whole number by a unit fraction, and compute such quotients.

Visual models for division of whole numbers by unit fractions and unit fractions by whole numbers

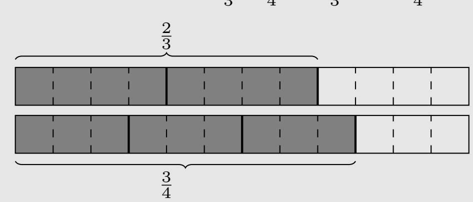


Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.



Reasoning with a fraction strip using the sharing interpretation of division: the strip is the whole and the shaded area is $\frac{1}{2}$ of the whole. If the shaded area is divided into 3 equal parts, then 2×3 of those parts make up the whole, so $\frac{1}{2} \div 3 = \frac{1}{2 \times 3} = \frac{1}{6}$.

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{2}{3} = ? \times \frac{3}{4}$



We find a common unit for comparing $\frac{2}{3}$ and $\frac{3}{4}$ by dividing each $\frac{1}{3}$ into 4 parts and each $\frac{1}{4}$ into 3 parts. Then $\frac{2}{3}$ is 8 parts when $\frac{3}{4}$ is divided into 9 equal parts, so $\frac{2}{3} = \frac{8}{9} \times \frac{3}{4}$, which is the same as saying that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

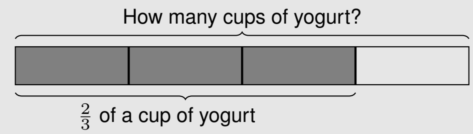
first factor:

$$\frac{2}{3} = ? \times \frac{3}{4}.$$

The same division problem can be interpreted using the sharing interpretation of division: how many cups are in a full container of yogurt when $\frac{2}{3}$ of a cup fills $\frac{3}{4}$ of the container. In other words, $\frac{3}{4}$ of what amount is equal to $\frac{2}{3}$ cups? In this case, $\frac{2}{3} \div \frac{3}{4}$ is seen as the solution to a multiplication equation with an unknown as the second factor:

$$\frac{3}{4} \times ? = \frac{2}{3}.$$

Visual model for $\frac{2}{3} \div \frac{3}{4}$ and $\frac{3}{4} \times ? = \frac{2}{3}$



The shaded area is $\frac{3}{4}$ of the entire strip. So it is 3 parts of a division of the strip into 4 equal parts. Another way of seeing this is that the strip is 4 parts of a division of the shaded area into 3 equal parts. That is, the strip is $\frac{4}{3}$ times the shaded part. So $? = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$.

There is a close connection between the reasoning shown in the margin and reasoning about ratios; if two quantities are in the ratio $3 : 4$, then there is a common unit so that the first quantity is 3 units and the second quantity is 4 units. The corresponding unit rate is $\frac{3}{4}$, and the first quantity is $\frac{3}{4}$ times the second quantity. Viewing the situation the other way around, with the roles of the two quantities interchanged, the same reasoning shows that the second quantity is $\frac{4}{3}$ times the first quantity. Notice that this leads us directly to the invert-and-multiply for fraction division: we have just reasoned that the ? in the equation above must be equal to $\frac{4}{3} \times \frac{2}{3}$, which is exactly what the rules gives us for $\frac{2}{3} \div \frac{4}{3}$.[6.NS.1]

The invert-and-multiply rule can also be understood algebraically as a consequence of the general rule for multiplication of fractions. If $\frac{a}{b} \div \frac{c}{d}$ is is defined to be the missing factor in the multiplication equation

$$? \times \frac{c}{d} = \frac{a}{b}$$

then the fraction that does the job is

$$? = \frac{ad}{bc},$$

as we can verify by putting it into the multiplication equation and using the already known rules of fraction multiplication and the properties of operations:

$$\frac{ad}{bc} \times \frac{c}{d} = \frac{(ad)c}{(bc)d} = \frac{a(cd)}{b(cd)} = \frac{a}{b} \times \frac{cd}{cd} = \frac{a}{b}.$$

6.NS.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

Compute fluently with multi-digit numbers and find common factors and multiples In Grade 6 students consolidate the work of earlier grades on operations with whole numbers and decimals by becoming fluent in the four operations on these numbers.[6.NS.2, 6.NS.3] Much of the foundation for this fluency has been laid in earlier grades. They have known since Grade 3 that whole numbers are fractions[3.NF.3c] and since Grade 4 that decimal notation is a way of writing fractions with denominator equal to a power of 10;[4.NF.6] by Grade 6 they start to see whole numbers, decimals and fractions

6.NS.2 Fluently divide multi-digit numbers using the standard algorithm.

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

3.NF.3c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

not as wholly different types of numbers but as as part of the same number system, represented by the number line.

In many traditional treatments of fractions greatest common factors occur in reducing a fraction to lowest terms, and least common multiples occur in adding fractions. As explained in the Fractions Progression, neither of these activities is treated as a separate topic in the standards. Indeed, insisting that finding a least common multiple is an essential part of adding fractions can get in the way of understanding the operation, and the excursion into prime factorization and factor trees that is often entailed in these topics can be time-consuming and distract from the focus of K–5. In Grade 6, however, students experience a modest introduction to the concepts[6.NS.4] and put the idea of greatest common factor to use in a rehearsal for algebra, where they will need to see, for example, that $3x^2 + 6x = 3x(x + 2)$.

[6.NS.4] Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor.

Apply and extend previous understandings of numbers to the system of rational numbers In Grade 6 the number line is extended to include negative numbers. Students initially encounter negative numbers in contexts where it is natural to describe both the magnitude of the quantity, e.g. vertical distance from sea level in meters, and the direction of the quantity (above or below sea level).[6.NS.5] In some cases 0 has an essential meaning, for example that you are at sea level; in other cases the choice of 0 is merely a convention, for example the temperature designated as 0° in Farenheit or Celsius. Although negative integers might be commonly used as initial examples of negative numbers, the Standards do not introduce the integers separately from the entire system of rational numbers, and examples of negative fractions or decimals can be included from the beginning.

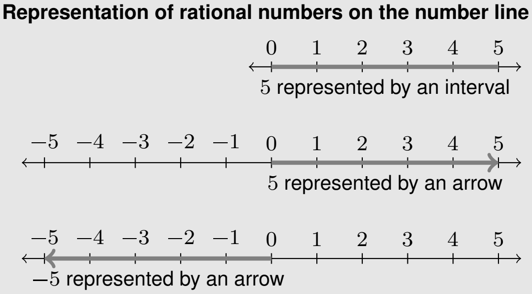
[6.NS.5] Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

[6.NS.6] Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

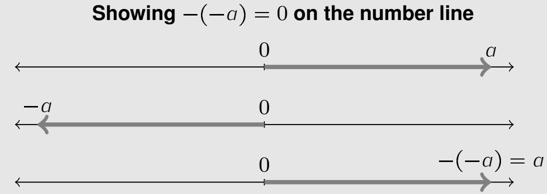
a Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

Directed measurement scales for temperature and elevation provide a basis for understanding positive and negative numbers as having a positive or negative direction on the number line.[6.NS.6a] Previous understanding of numbers on the number line related the position of the number to measurement: the number 5 is located at the endpoint of an line segment 5 units long whose other endpoint is at 0. Now the line segments acquire direction; starting at 0 they can go in either the positive or the negative direction. These directed numbers can be represented by putting arrows at the endpoints of the line segments.

Representation of rational numbers on the number line



Showing $-(-a) = 0$ on the number line



Students come to see $-p$ as the opposite of p, located an equal distance from 0 in the opposite direction. In order to avoid the common misconception later in algebra that any symbol with a negative sign in front of it should be a negative number, it is useful for students to see examples where $-p$ is a positive number, for example if $p = -3$ then $-p = -(-3) = 3$. Students come to see the operation of putting a negative sign in front of a number as flipping the direction of the number from positive to negative or negative to positive. Students generalize this understanding of the meaning of the negative sign to the coordinate plane, and can use it in locating numbers on the number line and ordered pairs in the coordinate plane.[6.NS.6bc]

With the introduction of negative numbers, students gain a new sense of ordering on the number line. Whereas statements like $5 < 7$ could be understood in terms of having more of or less of a certain quantity—"I have 5 apples and you have 7, so I have fewer than you"—comparing negative numbers requires closer attention to the relative positions of the numbers on the number line rather than their magnitudes.[6.MS.7a] Comparisons such as $-7 < -5$ can initially be confusing to students, because -7 is further away from 0 than -5, and is therefore larger in magnitude. Referring back to contexts in which negative numbers were introduced can be helpful: 7 meters below sea level is lower than 5 meters below sea level, and $-7°$ F is colder than $-5°$ F. Students are used to thinking of colder temperatures as lower than hotter temperatures, and so the mathematically correct statement also makes sense in terms of the context.[6.NS.7b]

At the same time, the prior notion of distance from 0 as a measure of size is still present in the notion of absolute value. To avoid confusion it can help to present students with contexts where it makes sense both to compare the order of two rational numbers and to compare their absolute value, and where these two comparisons run in different directions. For example, someone with a balance of \$100 in their bank account has more money than someone with a balance of $-\$1000$, because $100 > -1000$. But the second person's debt is much larger than the first person's credit $|-1000| > |100|$.[6.NS.7cd]

This understanding is reinforced by extension to the coordinate plane.[6.NS.8]

b Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

6.NS.7 Understand ordering and absolute value of rational numbers.

a Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram.

b Write, interpret, and explain statements of order for rational numbers in real-world contexts.

c Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.

d Distinguish comparisons of absolute value from statements about order.

6.NS.8 Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Grade 7

Addition and subtraction of rational numbers In Grade 6 students learned to locate rational numbers on the number line; in Grade 7 they extend their understanding of operations with fractions to operations with rational numbers. Whereas previously addition was represented by concatenating the line segments together, now the the line segments have directions, and therefore a beginning and an end. When concatenating these directed line segments, we start the second line segment at the end of the first one. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively cancelling part or all of it out.[7.NS.1b] Later in high school, if students encounter vectors, they will be able to see this as one-dimensional vector addition.

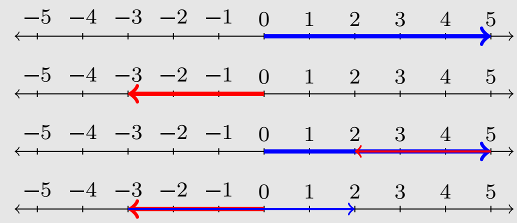
A fundamental fact about addition of rational numbers is that $p + (-p) = 0$ for any rational number p; in fact, this is a new property of operations that comes into play when negative numbers are introduced. This property can be introduced using situations in which the equation makes sense in a context.[7.NS.1a] For example, the operation of adding an integer could be modeled by an elevator moving up or down a certain number of floors. It can also be shown using the directed line segment model of addition on the number, as shown in the margin.[7.NS.1b]

It is common to use colored chips to represent integers, with one color representing positive integers and another representing negative integers, subject to the rule that chips of different colors cancel each other out; thus, a number is not changed if you take away or add such a pair. This is rather a different representation than the number line. On the number line, the equation $p+(-p) = 0$ follows from the definition of addition using directed line segments. With integer chips, the equation $p + (-p) = 0$ is true by definition since it it is encoded in the rules for manipulating the chips. Also implicit in the use of chips is that the commutative and associative properties extend to addition of integers, since combining chips can be done in any order.

However, the integer chips are not suited to representing rational numbers that are not integers. Whether such chips are used or not, the Standards require that students eventually understand location and addition of rational numbers on the number line. With the number line model, showing that the properties of operations extend to rational numbers requires some reasoning. Although it is not appropriate in Grade 6 to insist that all the properties be proved proved to hold in the number line or chips model, experimenting with them in these models is a good venue for reasoning (MP.2).[7.NS.1d]

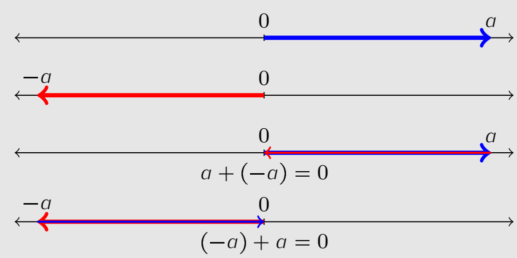
Subtraction of rational numbers is defined the same way as for positive rational numbers: $p-q$ is defined to be the missing addend in $q + ? = p$. For example, in earlier grades, students understand $5 - 3$ as the missing addend in $3 + ? = 5$. On the number line, it

Showing $5 + (-3) = 2$ and $-3 + 5 = 2$ on the number line



The number 5 is represented by the blue arrow pointing right from 0, and the number -3 is represented by the red arrow pointing left from 0. To add $5 + (-3)$ we place the arrow for 5 down first then attach the arrow for -3 to its endpoint. To add $-3 + 5$ we place the arrow for -3 down first then attach the arrow for 5 to its endpoint.

Showing $a + (-a) = 0$, and $(-a) + a = 0$ on the number line



7.NS.1 Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

- a Describe situations in which opposite quantities combine to make 0.
- b Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.
- d Apply properties of operations as strategies to add and subtract rational numbers.

is represented as the distance from 3 to 5. Or, with our newfound emphasis on direction on the number line, we might say that it is how you get from 3 from 5; by going two units to the right (that is, by adding 2).

In Grade 6 students apply the same understanding to $(-5) - (-3)$. It is the missing addend in $(-3) + ? = -5$, or how you get from -3 to -5. Since -5 is two units to the left of -3 on the number line, the missing addend is -2.

With the introduction of direction on the number line, there is a distinction between the distance from a and b and how you get from a to b. The distance from -3 to -5 is 2 units, but the instructions how to get from -3 to -5 are "go two units to the left." The distance is a positive number, 2, wherease "how to get there" is a negative number -2. In Grade 6 we introduce the idea of absolute value to talk about the size of a number, regardless of its sign. It is always a positive number or zero. If p is positive, then its absolute value $|p|$ is just p; if p is negative then $|p| = -p$. With this interpretation we can say that the absolute value of $p - q$ is just the distance from p to q, regardless of direction.[7.NS.1c]

Understanding $p - q$ as a missing addend also helps us see that $p + (-q) = p - q$.[7.NS.1c] Indeed, $p - q$ is the missing number in

$$q + ? = p$$

and $p + (-q)$ satisfies the description of being that missing, number:

$$q + (p + (-q)) = p + (q + (-q)) = p + 0 = p.$$

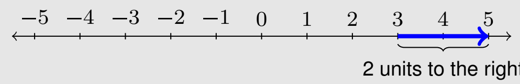
The figure in the margin illustrates this in the case where p and q are positive and $p > q$.

Multiplication and division of rational numbers Hitherto we have been able to come up with visual models to represent rational numbers, and the operations of addition and subtraction on them. This starts to break down with multiplication and division, and students must rely increasingly on the properties of operations to build the necessary bridges from their previous understandings to situations where one or more of the numbers might be negative.

For example, multiplication of a negative number by a positive whole number can still be understood as before; just as 5×2 can be understood as $2 + 2 + 2 + 2 + 2 = 10$, so 5×-2 can be understood as $(-2) + (-2) + (-2) + (-2) + (-2) = -10$. We think of 5×2 as five jumps to the right on the number line, starting at 0, and we think of $5 \times (-2)$ as five jumps to the left.

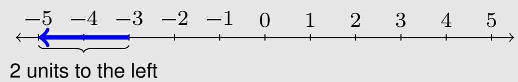
But what about $\frac{3}{4} \times -2$, or -5×-2? Perhaps the former can be undersood as going $\frac{3}{4}$ of the way from 0 to -2, that is $-\frac{3}{2}$. For the latter, teachers sometimes come up with metaphors involving going backwards in time or repaying debts. But in the end these metaphors do not explain why $-5 \times -2 = 10$. In fact, this is a

Showing $5 - 3 = 2$ on the number line.



You get from 3 to 5 by adding 2, so $5 - 3 = 2$.

Showing $(-5) - (-3) = -2$ on the number line.

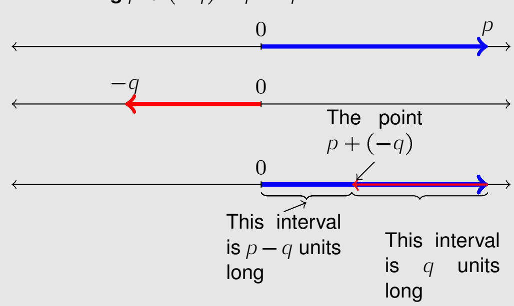


You get from -3 to -5 by adding -2, so $(-5) - (-3) = -2$

7.NS.1c Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

c Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Showing $p + (-q) = p - q$ on the number line



The red directed interval representing $-q$ is q units long, so the remaining part of the blue directed interval representing p is $p - q$ units long.

choice we make, not something we can justify by appeals to real world situations.

Why do we make the choice of saying that a negative times a negative is positive? Because we want to extend the operation of multiplication to rational number is such a way that *all* of the properties of operations are satisfied.[7.NS.2a] In particular, the property that really makes a difference here is the distributive property. If you want to be able to say that

$$4 \times (5 + (-2)) = 4 \times 5 + 4 \times (-2),$$

you have to say that $4 \times (-2) = -8$, because the number on the left is 12 and the first addend on the right is 20. This leads to the rules

$$\text{positive} \times \text{negative} = \text{negative} \quad \text{and} \quad \text{negative} \times \text{positive} = \text{negative}.$$

If you want to be able to say that

$$(-4) \times (5 + (-2)) = (-4) \times 5 + (-4) \times (-2),$$

then you have to say that $(-4) \times (-2) = 8$, since now we know that the number on the left is -12 and the first addend on the right is -20. This leads to the rule

$$\text{negative} \times \text{negative} = \text{positive}.$$

Why is it important to maintain the distributive property? Because when students get to algebra, they use it all the time. They must be able to say $-3x - 6y = -3(x + 2y)$ without worrying about whether x and y are positive or negative.

The rules about moving negative signs around in a product result from the rules about multiplying negative and positive numbers. Think about the various possibilities for p and q in

$$p \times (-q) = (-p) \times q = -pq.$$

If p and q are both positive, then this just a restatement of the rules above. But it still works if, for example, p is negative and q is positive. In that case it says

$$\text{negative} \times \text{negative} = \text{positive} \times \text{positive} = \text{positive}.$$

Just as the relationship between addition and subtraction helps students understand subtraction of rational numbers, so the relationship between multiplication and division helps them understand division. To calculate $-8 \div 4$, students recall that $(-2) \times 4 = -8$, and so $-8 \div 4 = -2$. By the same reasoning,

$$-8 \div 5 = -\frac{8}{5} \quad \text{because} \quad -\frac{8}{5} \times 5 = -8.$$

7.NS.2a Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

a Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

This means it makes sense to write

$$-8 \div 5 \quad \text{as} \quad \frac{-8}{5}.$$

Until this point students have not seen fractions where the numerator or denominator could be a negative integer. But working with the corresponding multiplication equations allows students to make sense of such fractions. In general, they see that[7.NS.2b]

$$-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$$

for any integers p and q, with $q \neq 0$.

Again, using multiplication as a guide, students can extend division to rational numbers that are not integers.[7.NS.2c] For example

$$\frac{2}{3} \div (-\frac{1}{2}) = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

And again it makes sense to write this division as a fraction:

$$\frac{\frac{2}{3}}{-\frac{1}{2}} = -\frac{4}{3} \quad \text{because} \quad -\frac{4}{3} \times -\frac{1}{2} = \frac{2}{3}.$$

Note that this is an extension of the fraction notation to a situation it was not originally designed for. There is no sense in which we can think of

$$\frac{\frac{2}{3}}{-\frac{1}{2}}$$

as $\frac{2}{3}$ parts where one part is obtained by dividing the line segment from 0 to 1 into $-\frac{1}{2}$ equal parts! But the fact that the properties of operations extend to rational numbers means that calculations with fractions extend to these so-called complex fractions $\frac{p}{q}$, where p and q could be rational numbers, not only integers. By the end of Grade 7, students are solving problems involving complex fractions.[7.NS.3]

Decimals are special fractions, those with denominator 10, 100, 1000, etc. But they can also be seen as a special sort of measurement on the number line, namely one that you get by partitioning into 10 equal pieces. In Grade 7 students begin to see this as a possibly infinite process. The number line is marked off into tenths, each of which is marked off into 10 hundreths, each of which is marked off into 10 thousandths, and so on ad infinitum. These finer and finer partitions constitute a sort of address system for numbers on the number line: 0.635 is, first, in the neighborhood betweeen 0.6 and 0.7, then in part of that neighborhood between 0.63 and 0.64, then exactly at 0.635.

The finite decimals are the rational numbers that eventually come to fall exactly on one of the tick marks in this decimal address system. But there are numbers that never come to rest, no

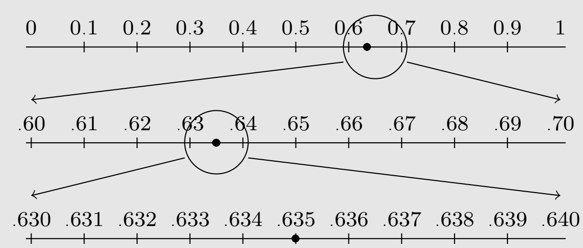
7.NS.2b Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

b Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts.

c Apply properties of operations as strategies to multiply and divide rational numbers.

7.NS.3 Solve real-world and mathematical problems involving the four operations with rational numbers.

Zoming in on 0.635



The finite decimal 0.635 can eventually be found sitting one of the tick marks at the thousandths level.

matter how far down you go. For example, $\frac{1}{3}$ is always sitting one third of the way along the third subdivision. It is 0.33 plus one-third of a thousandth, and 0.333 plus one-third of a ten thousandth, and so on. The decimals 0.33, 0.333, 0.3333 are successively closer and closer approximations to $\frac{1}{3}$. We summarize this situation by saying that $\frac{1}{3}$ has an infinite decimal expansion consisting entirely of 3s

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3},$$

where the bar over the 3 indicates that it repeats indefinitely. Although a rigorous treatment of this mysterious infinite expansion is not available in middle school, students in Grade 7 start to develop an intuitive understanding of decimals as a (possibly) infinite address system through simple examples such as this.[7.NS.2d]

For those rational numbers that have finite decimal expansions, students can find those expansions using long division. Saying that a rational number has a finite decimal expansion is the same as saying that it can be expressed as a fraction whose numerator is a base-ten unit (10, 100, 1000, etc.). So if $\frac{a}{b}$ is a fraction with a finite expansion, then

$$\frac{a}{b} = \frac{n}{10} \quad \text{or} \quad \frac{n}{100} \quad \text{or} \quad \frac{n}{1000} \quad \text{or} \quad \ldots,$$

for some whole number n. If this is the case, then

$$\frac{10a}{b} = n \quad \text{or} \quad \frac{100a}{b} = n \quad \text{or} \quad \frac{1000a}{b} = n \quad \text{or} \quad \ldots.$$

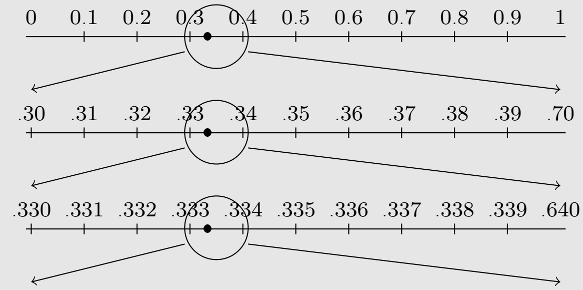
So we can find the whole number n by dividing b successively into $10a$, $100a$, $1000a$, and so on until there is no remainder.[7.NS.2d] The margin illustrates this process for $\frac{3}{8}$, where we find that there is no remainder for the division into 3000, so

$$3000 = 8 \times 375,$$

which means that

$$\frac{3}{8} = \frac{375}{1000} = 0.375.$$

Zooming in on $\frac{1}{3}$



The fraction $\frac{1}{3}$ is never eventually on one of the tick marks. It is always one third the way along the third subdivision.

7.NS.2d Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

d Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Division of 8 into 3 times a base-ten unit

$$\begin{array}{r} 3 \\ 8\,\overline{)\,30} \\ \underline{24} \\ 6 \end{array} \qquad \begin{array}{r} 37 \\ 8\,\overline{)\,300} \\ \underline{240} \\ 60 \\ \underline{56} \\ 4 \end{array} \qquad \begin{array}{r} 375 \\ 8\,\overline{)\,3000} \\ \underline{2400} \\ 600 \\ \underline{560} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Notice that it is not really necessary to restart the division for each new base-ten unit, since the steps from the previous calculation carry over to the next one.

Grade 8

Know that there are numbers that are not rational, and approximate them by rational numbers In Grade 7 students encountered infinitely repeating decimals, such as $\frac{1}{3} = 0.\overline{3}$. In Grade 8 they understand why this phenomenon occurs, a good exercise in expressing regularity in repeated reasoning (MP8).[8.NS.1] Taking the case of $\frac{1}{3}$, for example, we can try to express it as a finite decimal using the same process we used for $\frac{3}{8}$ in Grade 7. We successively divide 3 into 10, 100, 1000, hoping to find a point at which the remainder is zero. But this never happens; there is always a remainder of 1. After a few tries it is clear that the long division will always go the same way, because the individual steps always work the same way: the next digit in the quotient is always 3 resulting in a reduction of the dividend from one base-unit to the next smaller one (see margin). Once we have seen this regularity, we see that $\frac{1}{3}$ can never be a whole number of decimal base-ten units, no matter how small they are.

A similar investigation with other fractions leads to the insight that there must always eventually be a repeating pattern, because there are only so many ways a step in the algorithm can go. For example, considering the possibility that $\frac{2}{7}$ might be a finite decimal with, we try dividing 7 into 20, 200, 2000, etc., hoping to find a point where the remainder is zero. But something happens when we get to dividing 7 into 2,000,000, the left-most division in the margin. We find ourselves with a remainder of 2. Since we started with a numerator of 2, the algorithm is going to start repeating the sequence of digits from this point on. So we are never going to get a remainder of 0. All is not in vain, however. Each calculation gives us a decimal approximation of $\frac{2}{7}$. For example, the left-most calculation in the margin tells us that

$$\frac{2}{7} = \frac{1}{1000000}\frac{2000000}{7} = 0.285714 + \frac{2}{7} \times 0.0000001,$$

and the next two show that

$$\begin{aligned} \frac{2}{7} &= 0.2857142 + \frac{6}{7} \times 0.00000001 \\ \frac{2}{7} &= 0.28571428 + \frac{4}{7} \times 0.000000001. \end{aligned}$$

Noticing the emergence of the repeating pattern 285714 in the digits, we say that

$$\frac{2}{7} = 0.\overline{285714}.$$

The possibility of infinite repeating decimals raises the possibility of infinite decimals that do not ever repeat. From the point of view of the decimal address system, there is no reason why such number should not correspond to a point on the number line. For

[8.NS.1] Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Division of 3 into 100, 1000, and $10{,}000$

$$\begin{array}{r} 33 \\ 3\overline{)100} \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \qquad \begin{array}{r} 333 \\ 3\overline{)1000} \\ \underline{900} \\ 100 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \qquad \begin{array}{r} 3333 \\ 3\overline{)10000} \\ \underline{9000} \\ 1000 \\ \underline{900} \\ 100 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

Repeated division of 3 into larger and larger base ten units shows the same pattern.

Division of 7 into multiples of 2 times larger and larger base-ten units

$$\begin{array}{r} 285714 \\ 7\overline{)2000000} \\ \underline{1400000} \\ 600000 \\ \underline{560000} \\ 40000 \\ \underline{35000} \\ 5000 \\ \underline{4900} \\ 100 \\ \underline{70} \\ 30 \\ \underline{28} \\ 2 \end{array} \qquad \begin{array}{r} 2857142 \\ 7\overline{)20000000} \\ \underline{14000000} \\ 6000000 \\ \underline{5600000} \\ 400000 \\ \underline{350000} \\ 50000 \\ \underline{49000} \\ 1000 \\ \underline{700} \\ 300 \\ \underline{280} \\ 20 \\ \underline{14} \\ 6 \end{array} \qquad \begin{array}{r} 28571428 \\ 7\overline{)200000000} \\ \underline{140000000} \\ 60000000 \\ \underline{56000000} \\ 4000000 \\ \underline{3500000} \\ 500000 \\ \underline{490000} \\ 10000 \\ \underline{7000} \\ 3000 \\ \underline{2800} \\ 200 \\ \underline{140} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

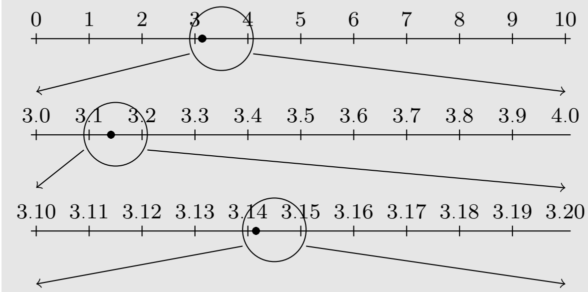
The remainder at each step is always a single digit multiple of a base-ten unit so eventually the algorithm has to cycle back to the same situation as some earlier step. From then on the algorithm produces the same sequence of digits as from the earlier step, ad infinitum.

example, the number π lives between 3 and 4, and between 3.1 and 3.2, and between 3.14 and 3.15, and so on, with each successive decimal digit narrowing its possible location by a factor of 10.

Numbers like π, which do not have a repeating decimal expansion and therefore are not rational numbers, are called *irrational*.[8.NS.1] Although we can calculate the decimal expansion of π to any desired accuracy, we cannot describe the entire expansion because it is infinitely long, and because there is no pattern (as far as we know). However, because of the way in which the decimal address system narrows down the interval in which a number lives, we can use the first few digits of the decimal expansion to come up with good decimal approximations of π, or any other irrational number. For example, the fact that π is between 3 and 4 tells us that π^2 is between 9 and 16; the fact that π is between 3.1 and 3.2 tells us that π^2 is between 9.6 and 10.3, and so on.[8.NS.2]

8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

Zooming in on π



The number π has an infinite non-repeating decimal expansion which determines each successive sub-interval to zoom in on.

8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2).

High School, Number*

The Real Number System

Extend the properties of exponents to rational exponents In Grades 6–8 students began to widen the possible types of number they can conceptualize on the number line. In Grade 8 they glimpse the existence of irrational numbers such as $\sqrt{2}$. In high school, they start a systematic study of functions that can take on irrational values, such as exponential, logarithmic, and power functions. The first step in this direction is the understanding of numerical expressions in which the exponent is not a whole number. Functions such as $f(x) = x^2$, or more generally polynomial functions, have the property that if the input x is a rational number, then so is the output. This is because their output values are computed by basic arithmetic operations on their input values. But a function such as $f(x) = \sqrt{x}$ does not necessarily have rational output values for every rational input value. For example, $f(2) = \sqrt{2}$ is irrational even though 2 is rational.

The study of such functions brings with it a need for an extended understanding of the meaning of an exponent. Exponent notation is a remarkable success story in the expansion of mathematical ideas. It is not obvious at first that a number such as $\sqrt{2}$ can be represented as a power of 2. But reflecting that

$$\left(\sqrt{2}\right)^2 = 2$$

and thinking about the properties of exponents, it is natural to define

$$2^{\frac{1}{2}} = \sqrt{2}$$

since if we follow the rule $(a^b)^c = a^{bc}$ then

$$\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2.$$

Similar reasoning leads to a general definition of the meaning of a^b whenever a and b are rational numbers.[N-RN.1] It should be noted high school mathematics does not develop the mathematical ideas necessary to prove that numbers such as $\sqrt{2}$ and $3^{\frac{1}{5}}$ actually exist;

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

*This progression concerns Number and Quantity standards related to number. The remaining standards are discussed in the Quantity Progression.

in fact all of high school mathematics depends on the fundamental assumption that properties of rational numbers extend to irrational numbers. This is not unreasonable, since the number line is populated densely with rational numbers, and a conception of number as a point on the number line gives reassurance from intuitions about measurement that irrational numbers are not going to behave in a strangely different way from rational numbers.

Because rational exponents have been introduced in such a way as to preserve the laws of exponents, students can now use those laws in a wider variety of situations. For example, they can rewrite the formula for the volume of a sphere of radius r,

$$V = \frac{4}{3}\pi r^3,$$

to express the radius in terms of the volume,[N-RN.2]

$$r = \left(\frac{3}{4}\frac{V}{\pi}\right)^{\frac{1}{3}}.$$

N-RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers An important difference between rational and irrational numbers is that rational numbers form a number system. If you add, subtract, multiply, or divide two rational numbers, you get another rational number (provided the divisor is not 0 in the last case). The same is not true of irrational numbers. For example, if you multiply the irrational number $\sqrt{2}$ by itself, you get the rational number 2. Irrational numbers are defined by not being rational, and this definition can be exploited to generate many examples of irrational numbers from just a few.[N-RN.3] For example, because $\sqrt{2}$ is irrational it follows that $3+\sqrt{2}$ and $5\sqrt{2}$ are also irrational. Indeed, if $3+\sqrt{2}$ were an irrational number, call it x, say, then from $3+\sqrt{2} = x$ we would deduce $\sqrt{2} = x-3$. This would imply $\sqrt{2}$ is rational, since it is obtained by subtracting the rational number 3 from the rational number x. But it is not rational, so neither is $3+\sqrt{2}$.

N-RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Although in applications of mathematics the distinction between rational and irrational numbers is irrelevant, since we always deal with finite decimal approximations (and therefore with rational numbers), thinking about the properties of rational and irrational numbers is good practice for mathematical reasoning habits such as constructing viable arguments and attending to precisions (MP.3, MP.6).

Complex Numbers

That complex numbers have a practical application is surprising to many. But it turns out that many phenomena involving real numbers become simpler when the real numbers are viewed as a subsytem of the complex numbers. For example, complex solutions of differential equations can give a unified picture of the behavior of real solutions. Students get a glimpse of this when they study complex solutions of quadratic equations. When complex numbers are brought into the picture, every quadratic polynomial can be expressed as a product of linear factors:

$$ax^2 + bx + c = a(x - r)(x - s).$$

The roots r and s are given by the quadratic formula:

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

When students first apply the quadratic formula to quadratic equations with real coefficients, the square root is a problem if the quantity $b^2 - 4ac$ is negative. Complex numbers solve that problem by introducing a new number, i, with the property that $i^2 = -1$, which enables students to express the solutions of any quadratic equation.[N-CN.7]

N-CN.7 Solve quadratic equations with real coefficients that have complex solutions.

One remarkable fact about introducing the number i is that it works: the set of numbers of the form $a + bi$, where $i^2 = -1$ and a and b are real numbers, forms a number system. That is, you can add, subtract, multiply and divide two numbers of this form and get another number of the same form as the result. We call this the system of complex numbers.[N-CN.1]

N-CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

All you need to perform operations on complex numbers is the fact that $i^2 = -1$ and the properties of operations.[N-CN.2] For example, to add $3 + 2i$ and $-1 + 4i$ we write

N-CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

$$(3 + 2i) + (-1 + 4i) = (3 + -1) + (2i + 4i) = 2 + 6i,$$

using the associative and commutative properties of addition, and the distributive property to pull the i out, resulting in another complex number. Multiplication requires using the fact that $i^2 = -1$:

$$(3 + 2i)(-1 + 4i) = -3 + 10i + 8i^2 = -3 + 10i - 8 = -11 + 10i.$$

Division of complex numbers is a little tricker, but with the discovery of the complex conjugate $a - bi$ we find that every non-zero complex number has a multiplicative inverse.[N-CN.3] If at least one of a and b is not zero, then

N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

$$(a + bi)^{-1} = \frac{1}{a^2 + b^2}(a - bi)$$

because

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

Students who continue to study geometric representations of complex numbers in the complex plane use both rectangular and polar coordinates which leads to a useful geometric interpretation of the operations.[N-CN.4, N-CN.5] The restriction of these geometric interpretations to the real numbers yields and interpretation of these operations on the number line.

One of the great theorems of modern mathematics is the Fundamental Theorem of Algebra, which says that every polynomial equation has a solution in the complex numbers. To put this into perspective, recall that we formed the complex numbers by creating a solution, i, to just one special polynomial equation, $x^2 = -1$. With the addition of this one solution, it turns out that every polynomial equation, for example $x^4 + x^2 = -1$, also acquires a solution. Students have already seen this phenomenon for quadratic equations.[N-CN.9]

Although much of the study of complex numbers goes beyond the college and career ready threshold, as indicated by the (+) on many of the standards, it is a rewarding area of exploration for advanced students.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

N-CN.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Progressions for the Common Core State Standards in Mathematics (draft)

2 July 2013

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For updates and more information about the Progressions, see `http://ime.math.arizona.edu/progressions`.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: `http://commoncoretools.me`.

Grade 8, High School, Functions*

Overview

Functions describe situations in which one quantity is determined by another. The area of a circle, for example, is a function of its radius. When describing relationships between quantities, the defining characteristic of a *function* is that the input value determines the output value or, equivalently, that the output value depends upon the input value.

The mathematical meaning of function is quite different from some common uses of the word, as in, "One function of the liver is to remove toxins from the body," or "The party will be held in the function room at the community center." The mathematical meaning of function is close, however, to some uses in everyday language. For example, a teacher might say, "Your grade in this class is a function of the effort you put into it." A doctor might say, "Some illnesses are a function of stress." Or a meteorologist might say, "After a volcano eruption, the path of the ash plume is a function of wind and weather." In these examples, the meaning of "function" is close to its mathematical meaning.

In some situations where two quantities are related, each can be viewed as a function of the other. For example, in the context of rectangles of fixed perimeter, the length can be viewed as depending upon the width or vice versa. In some of these cases, a problem context may suggest which one quantity to choose as the input variable.

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*The study of functions occupies a large part of a student's high school career, and this document does not treat in detail all of the material studied. Rather it gives some general guidance about ways to treat the material and ways to tie it together. It notes key connections among standards, points out cognitive difficulties and pedagogical solutions, and gives more detail on particularly knotty areas of the mathematics.

The high school standards specify the mathematics that all students should study in order to be college and career ready. Additional material corresponding to (+) standards, mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics, is indicated by plus signs in the left margin.

Undergraduate mathematics may involve functions of more than one variable. The area of a rectangle, for example, can be viewed as a function of two variables: its width and length. But in high school mathematics the study of functions focuses primarily on real-valued functions of a single real variable, which is to say that both the input and output values are real numbers. One exception is in high school geometry, where geometric transformations are considered to be functions.• For example, a translation T, which moves the plane 3 units to the right and 2 units up might be represented by $T : (x, y) \mapsto (x + 3, y + 2)$.

• G-CO.2 . . . [D]escribe transformations as functions that take points in the plane as inputs and give other points as outputs. . . .

The problem with patterns

Students are asked to continue the pattern 2, 4, 6, 8, ... Here are some legitimate responses:

- Cody: I am thinking of a "plus 2 pattern," so it continues 10, 12, 14, 16, ...
- Ali: I am thinking of a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, ...
- Suri: I am thinking of the units digit in the multiples of 2, so it continues 0, 2, 4, 6, 8, 0, 2, ...
- Erica: If $g(n)$ is any polynomial, then $f(n) = 2n + (n-1)(n-2)(n-3)(n-4)g(n)$ describes a continuation of this sequence.
- Zach: I am thinking of that high school cheer, "Who do we appreciate?"

Because the task provides no structure, all of these answers must be considered correct. Without any structure, continuing the pattern is simply speculation—a guessing game. Because there are infinitely many ways to continue a sequence, patterning problems should provide enough structure so that the sequence is well defined.

Sequences and functions Patterns are sequences, and sequences are functions with a domain consisting of whole numbers. However, in many elementary patterning activities, the input values are not given explicitly. In high school, students learn to use an index to indicate which term is being discussed. In the example in the margin, Erica handles this issue by deciding that the term 2 would correspond to an index value of 1. Then the terms 4, 6, and 8 would correspond to input values of 2, 3, and 4, respectively. Erica could have decided that the term 2 would correspond to a different index value, such as 0. The resulting formula would have been different, but the (unindexed) sequence would have been the same.

Functions and Modeling In modeling situations, knowledge of the context and statistics are sometimes used together to find a function defined by an algebraic expression that best fits an observed relationship between quantities. (Here "best" is assessed informally, see the Modeling Progression and high school Statistics and Probability Progression for further discussion and examples.) Then the algebraic expressions can be used to interpolate (i.e., approximate or predict function values between and among the collected data values) and to extrapolate (i.e., to approximate or predict function values beyond the collected data values). One must always ask whether such approximations are reasonable in the context.

In school mathematics, functional relationships are often given by algebraic expressions. For example, $f(n) = n^2$ for $n \geqslant 1$ gives the n^{th} square number. But in many modeling situations, such as the temperature at Boston's Logan Airport as a function of time, algebraic expressions may not be suitable.

Functions and Algebra See the Algebra Progression for a discussion of the connection and distinctions between functions, on the one hand, and algebra and equation solving, on the other. Perhaps the most productive connection is that solving equations can be seen as finding the intersections of graphs of functions.[A-REI.11]

A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

K–7 foundations for functions Before they learn the term "function," students begin to gain experience with functions in elementary grades. In Kindergarten, they use patterns with numbers such as the $5+n$ pattern to learn particular additions and subtractions.

A trickle of pattern standards in Grades 4 and 5 continues the preparation for functions.[4.OA.5, 5.OA.3] Note that in both these standards a rule is explicitly given. Traditional pattern activities, where students are asked to continue a pattern through observation, are not a mathematical topic, and do not appear in the Standards in their own right.[1]

The Grade 4–5 pattern standards expand to the domain of Ratios and Proportional Relationships in Grades 6–7. In Grade 6, as they work with collections of equivalent ratios, students gain experience with tables and graphs, and correspondences between them. They attend to numerical regularities in table entries and corresponding geometrical regularities in their graphical representations as plotted points.[MP.8] In Grade 7, students recognize and represent an important type of regularity in these numerical tables—the multiplicative relationship between each pair of values—by equations of the form $y = cx$, identifying c as the constant of proportionality in equations and other representations[7.RP.2] (see the Ratios and Proportional Relationships Progression).

The notion of a function is introduced in Grade 8. Linear functions are a major focus, but note that students are also expected to give examples of functions that are not linear.[8.F.3] In high school, students deepen their understanding of the notion of function, expanding their repertoire to include quadratic and exponential functions, and increasing their understanding of correspondences between geometric transformations of graphs of functions and algebraic transformations of the associated equations.[F-BF.3] The trigonometric functions are another important class of functions. In high school, students study trigonometric ratios in right triangles.[G-SRT.6] Understanding radian measure of an angle as arc length on the unit circle enables students to build on their understanding of trigonometric ratios associated with acute angles, and to explain how these ratios extend to trigonometric functions whose domains are included in the real numbers.

The (+) standards for the conceptual categories of Geometry and Functions detail further trigonometry addressed to students who intend to take advanced mathematics courses such as calculus. This includes the Law of Sines and Law of Cosines, as well as further study of the values and properties of trigonometric functions.

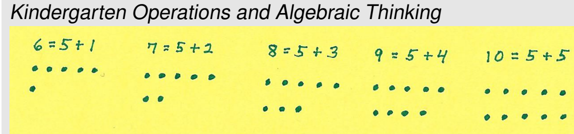
[1] This does not exclude activities where patterns are used to support other standards, as long as the case can be made that they do so.

4.OA.5 Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

5.OA.3 Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.

Experiences with functions before Grade 8

Kindergarten Operations and Algebraic Thinking



$f(n) = 5 + n$

Grade 3 Operations and Algebraic Thinking

$1 \times 9 = 9$

$2 \times 9 = 2 \times (10-1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$

$3 \times 9 = 3 \times (10-1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$

$f(n) = 9 \times n = 10 \times n - n$

Grade 4 Geometric Measurement

feet	0	1	2	3	
inches	0	12	24		

$f(t) = 12t$

Grade 6 Ratios and Proportional Relationships

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

$f(t) = \frac{3}{2}t$

MP.8 "Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts."

F-BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

Grade 8

Define, evaluate, and compare functions Since the elementary grades, students have been describing patterns and expressing relationships between quantities. These ideas become semi-formal in Grade 8 with the introduction of the concept of *function*: a rule that assigns to each input exactly one output.[8.F.1] Formal language, such as domain and range, and function notation may be postponed until high school.

Building on their earlier experiences with graphs and tables in Grades 6 and 7, students a routine of exploring functional relationships algebraically, graphically, numerically in tables, and through verbal descriptions.[8.F.2] They explain correspondences between equations, verbal descriptions, tables, and graphs (MP.1). Repeated reasoning about entries in tables or points on graphs results in equations for functional relationships (MP.8). To develop flexibility in interpreting and translating among these various representations, students compare two functions represented in different ways, as illustrated by "Battery Charging" in the margin.

The main focus in Grade 8 is linear functions, those of the form $y = mx + b$, where m and b are constants.[8.F.3] Students learn to recognize linearity in a table: when constant differences between input values produce constant differences between output values. And they can use the constant rate of change appropriately in a verbal description of a context.

The proof that $y = mx + b$ is also the equation of a line, and hence that the graph of a linear function is a line, is an important piece of reasoning connecting algebra with geometry in Grade 8. See the Expressions and Equations Progression.

Connection to Algebra and Geometry In high school, after students have become fluent with geometric transformations and have worked with similarity, another connection between algebra and geometry can be made in the context of linear functions.

The figure in the margin shows a "slope triangle" with one red side formed by the vertical intercept and the point on the line with x-coordinate equal to 1. The larger triangle is formed from the intercept and a point with arbitrary x-coordinate. A dilation with center at the vertical intercept and scale factor x takes the slope triangle to the larger triangle, because it takes lines to parallel lines.[G-SRT.1a] Thus the larger triangle is similar to the slope triangle,[G-SRT.2] and so the height of the larger triangle is mx, and the coordinates of the general point on the triangle are $(x, b + mx)$. Which is to say that the point satisfies the equation $y = b + mx$.

[8.F.1] Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.Function notation is not required in Grade 8.

[8.F.2] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

[MP.1] "Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends."

Battery Charging

Sam wants to take his MP3 player and his video game player on a car trip. An hour before they plan to leave, he realized that he forgot to charge the batteries last night. At that point, he plugged in both devices so they can charge as long as possible before they leave.

Sam knows that his MP3 player has 40% of its battery life left and that the battery charges by an additional 12 percentage points every 15 minutes.

His video game player is new, so Sam doesn't know how fast it is charging but he recorded the battery charge for the first 30 minutes after he plugged it in.

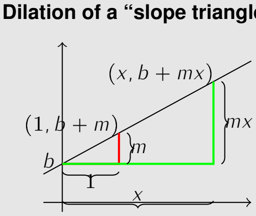
time charging in minutes	0	10	20	30
percent player battery charged	20	32	44	56

1. If Sam's family leaves as planned, what percent of the battery will be charged for each of the two devices when they leave?
2. How much time would Sam need to charge the battery 100% on both devices?

Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/641.

[8.F.3] Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

Dilation of a "slope triangle"



[G-SRT.1a] A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

[G-SRT.2] Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

Use functions to model relationships between quantities When using functions to model a linear relationship between quantities, students learn to determine the rate of change of the function, which is the slope of the line that is its graph. They can read (or compute or approximate) the rate of change from a table or a graph, and they can interpret the rate of change in context.[8.F.4]

Graphs are ubiquitous in the study of functions, but it is important to distinguish a function from its graph. For example, a linear function does not have a slope but the graph of a non-vertical line has a slope.•

Within the class of linear functions, students learn that some are proportional relationships and some are not. Functions of the form $y = mx + b$ are proportional relationships exactly when $b = 0$, so that y is proportional to x. Graphically, a linear function is a proportional relationship if its graph goes through the origin.

To understand relationships between quantities, it is often helpful to describe the relationships qualitatively, paying attention to the general shape of the graph without concern for specific numerical values.[8.F.5] The standard approach proceeds from left to right, describing what happens to the output as the input value increases. For example, pianist Chris Donnelly describes the relationship between creativity and structure via a graph.•

The qualitative description might be as follows: "As the input value (structure) increases, the output (creativity) increases quickly at first and gradually slowing down. As input (structure) continues to increase, the output (creativity) reaches a maximum and then starts decreasing, slowly at first, and gradually faster." Thus, from the graph alone, one can infer Donnelly's point that there is an optimal amount of structure that produces maximum creativity. With little structure or with too much structure, in contrast, creativity is low.

Connection to Statistics and Probability In Grade 8, students plot bivariate data in the coordinate plane (by hand or electronically) and use linear functions to analyze the relationship between two paired variables.[8.SP.2] See the Grades 6–8 Statistics and Probability Progression.

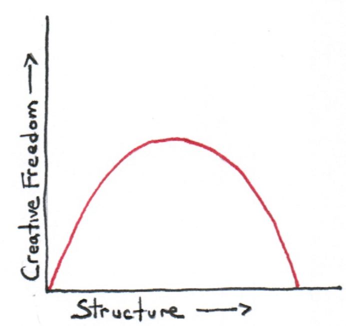
In high school, students take a deeper look at bivariate data, making use of their expanded repertoire of functions in modeling associations between two variables. See the sections on bivariate data and interpreting linear models in the High School Statistics and Probability Progression.

8.F.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

• The slope of a vertical line is undefined and the slope of a horizontal line is 0. Either of these cases might be considered "no slope." Thus, the phrase "no slope" should be avoided because it is ambiguous and "non-existent slope" and "slope of 0" should be distinguished from each other.

8.F.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

•



8.SP.2 Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

High School

The high school standards on functions are organized into four groups: Interpreting Functions (F-IF); Building Functions (F-BF); Linear, Quadratic and Exponential Models (F-LE); and Trigonometric Functions (F-TF). The organization of the first two groups under mathematical practices rather than types of function is an important aspect of the Standards: students should develop ways of thinking that are general and allow them to approach any type of function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary. For example, they should see linear and exponential functions as arising out of structurally similar growth principles; they should see quadratic, polynomial, and rational functions as belonging to the same system (helped along by the unified study in the Algebra category of Arithmetic with Polynomials and Rational Expressions).

Interpreting Functions

Understand the concept of a function and use function notation Building on semi-formal notions of functions from Grade 8, students in high school begin to use formal notation and language for functions. Now the input/output relationship is a correspondence between two sets: the domain and the range.[F-IF.1] The domain is the set of input values, and the range is the set of output values. A key advantage of function notation is that the correspondence is built into the notation. For example, $f(5)$ is shorthand for "the output value of f when the input value is 5."

Students sometimes interpret the parentheses in function notation as indicating multiplication. Because they might have seen numerical expressions like $3(4)$, meaning 3 times 4, students can interpret $f(x)$ as f times x. This can lead to false generalizations of the distributive property, such replacing $f(x+3)$ with $f(x)+f(3)$. Work with correspondences between values of the function represented in function notation and their location on the graph of f can help students avoid this misinterpretation of the symbols (see "Interpreting the Graph" in margin).

Although it is common to say "the function $f(x)$," the notation $f(x)$ refers to a single output value when the input value is x. To talk about the function as a whole, write f, or perhaps "the function f, where $f(x) = 3x+4$." The x is merely a placeholder, so $f(t) = 3t+4$ describes exactly the same function.

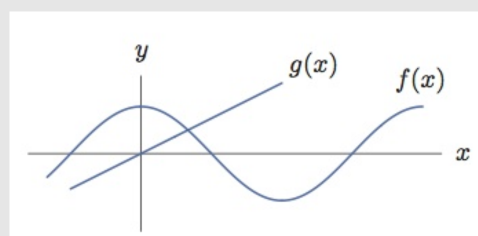
Later, students can make interpretations like those in the following table:

Expression	Interpretation
$f(a+2)$	The output when the input is 2 greater than a
$f(a)+3$	3 more than the output when the input is a
$2f(x)+5$	5 more than twice the output of f when the input is x
$f(b)-f(a)$	The change in output when the input changes from a to b

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

Interpreting the Graph

Use the graph (for example, by marking specific points) to illustrate the statements in (a)–(d). If possible, label the coordinates of any points you draw.



(a) $f(0) = 2$

(b) $f(-3) = f(3) = f(9) = 0$

(c) $f(2) = g(2)$

(d) $g(x) > f(x)$ for $x > 2$

Task from Illustrative Mathematics. For solutions and discussion, see at illustrativemathematics.org/illustrations/636.

MP.1 "Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs. . . ."

Notice that a common preoccupation of high school mathematics, distinguishing functions from relations, is not in the Standards. Time normally spent on exercises involving the vertical line test, or searching lists of ordered pairs to find two with the same x-coordinate and different y-coordinate, can be reallocated elsewhere. Indeed, the vertical line test is problematic, because it makes it difficult to discuss questions such as “Is x a function of y?” (an important question for students thinking about inverse functions) using a graph in which x-coordinates are on the horizontal axis. The essential question when investigating functions is: “Does each element of the domain correspond to exactly one element in the range?” The margin shows a discussion of the square root function oriented around this question.

The square root function

Since the equation $x^2 = 9$ has two solutions, $x = \pm 3$, students might think incorrectly that $\sqrt{9} = \pm 3$. However, if we want $\sqrt{x}$ to be a function of x, we need to choose one of these square roots. The square root function, $g(x) = \sqrt{x}$, is defined to be the positive square root of x for any positive x.

To promote fluency with function notation, students interpret function notation in contexts.F-IF.2 For example, if h is a function that relates Kristin’s height in inches to her age in years, then the statement $h(7) = 49$ means, “When Kristin was 7 years old, she was 49 inches tall.” The value of $h(12)$ is the answer to “How tall was Kristin when she was 12 years old.” And the solution of $h(x) = 60$ is the answer to “How old was Kristin when she was 60 inches tall?” See also “Cell Phones” in the margin.

F-IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MP.2 “Mathematically proficient students . . . [have] the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.”

Cell Phones

Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$

(b) $f(a) = 20$

(c) $f(20) = b$

(d) $n = f(t)$

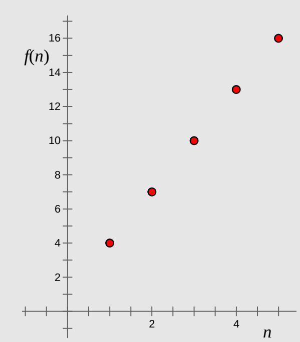
Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/634.

Sometimes, especially in real-world contexts, there is no expression (or closed formula) for a function. In those cases, it is common to use a graph or a table of values to (partially) represent the function.

A *sequence* is a function whose domain is a subset of the integers.F-IF.3 In fact, many patterns explored in grades K–8 can be considered sequences. For example, the sequence 4, 7, 10, 13, 16, … might be described as a “plus 3 pattern” because terms are computed by adding 3 to the previous term. To show how the sequence can be considered a function, we need an *index* that indicates which term of the sequence we are talking about, and which serves as an input value to the function. Deciding that the 4 corresponds to an index value of 1, we make a table showing the correspondence, as in the margin. The sequence can be described recursively by the rule $f(1) = 4$, $f(n+1) = f(n) + 3$ for $n \geqslant 2$. Notice that the recursive definition requires both a starting value and a rule for computing subsequent terms. The sequence can also be described with the closed formula $f(n) = 3n + 1$, for integers $n \geqslant 1$. Notice that the domain is included as part of the description. A graph of the sequence consists of discrete points, because the specification does not indicate what happens “between the dots.”

Sequences as functions

n	$f(n)$
1	4
2	7
3	10
4	13
5	16



F-IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

\+ In courses that address material corresponding to the plus standards, students may use subscript notation for sequences.

Interpret functions that arise in applications in terms of the context Functions are often described and understood in terms of their *behavior*.F-IF.4 Over what input values is it increasing, decreasing, or constant? For what input values is the output value positive,

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

negative, or 0? What happens to the output when the input value gets very large in magnitude? Graphs become very useful representations for understanding and comparing functions because these "behaviors" are often easy to see in the graphs of functions (see "Warming and Cooling" in the margin). Graphs and contexts are opportunities to talk about the notion of the domain of a function (for an illustration, go to illustrativemathematics.org/illustrations/631).[F-IF.5]

Graphs help us reason about rates of change. Students learned in Grade 8 that the *rate of change* of a linear function is equal to the slope of the line that is its graph.[8.EE.5] And because the slope of a line is constant, that is, between any two points it is the same[8.EE.6] (see the Expressions and Equations Progression), "*the* rate of change" has an unambiguous meaning for a linear function. For nonlinear functions, however, rates of change are not constant, and so we talk about average rates of change over an interval.[F-IF.6]

For example, for the function $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line from $(2, 4)$ to $(5, 25)$ on the graph of g. And if g is interpreted as returning the area of a square of side x, then this calculation means that over this interval the area changes, on average, 7 square units for each unit increase in the side length of the square.

Analyze functions using different representations Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of a function and its key features.[F-IF.7]

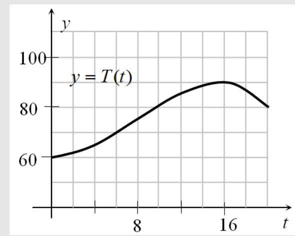
Within a family, the functions often have commonalities in the shapes of their graphs and in the kinds of features that are important for identifying functions more precisely within a family. This standard indicates which function families should be in students' repertoires, detailing which features are required for several key families. It is an overarching standard that covers the entire range of a student's high school experience; in this part of the progression we merely indicate some guidelines for how it should be treated.

First, linear and exponential functions (and to a lesser extent quadratic functions) receive extensive treatment and comparison in a dedicated group of standards, Linear and Exponential Models. Thus, those function families should receive the bulk of the attention related to this standard. Second, all students are expected to develop fluency with linear, quadratic, and exponential functions, including the ability to graph them by hand. Finally, in most of the other function families, students are expected to graph simple cases without technology, and more complex ones with technology.

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Warming and Cooling

The figure shows the graph of T, the temperature (in degrees Fahrenheit) over one particular 20-hour period in Santa Elena as a function of time t.



(a) Estimate $T(14)$.

(b) If $t = 0$ corresponds to midnight, interpret what we mean by $T(14)$ in words.

(c) Estimate the highest temperature during this period from the graph.

(d) When was the temperature decreasing?

(e) If Anya wants to go for a two-hour hike and return before the temperature gets over 80 degrees, when should she leave?

Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/639.

8.EE.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

8.EE.6 Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a Graph linear and quadratic functions and show intercepts, maxima, and minima.

b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

d (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Consistent with the practice of looking for and making use of structure (MP.7), students should also develop the practice of writing expressions for functions in ways that reveal the key features of the function.[F-IF.8]

Quadratic functions provide a rich playground for developing this ability, since the three principal forms for a quadratic expression (expanded, factored, and completed square) each give insight into different aspects of the function. However, there is a danger that working with these different forms becomes an exercise in picking numbers out of an expression. For example, students often arrive at college talking about "minus b over $2a$ method" for finding the vertex of the graph of a quadratic function. To avoid this problem it is useful to give students tasks such as "Which Expression?" in the margin, where they must read both the graphs and the expression and choose for themselves which parts of each correspond.[F-IF.9]

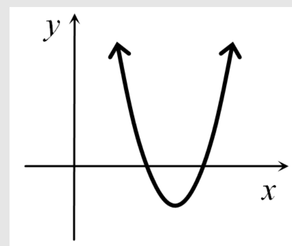
[F-IF.8] Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

- a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- b Use the properties of exponents to interpret expressions for exponential functions.

Which Expression?

Which of the following could be an expression for the function whose graph is shown below? Explain.

(a) $(x+12)^2+4$	(b) $-(x-2)^2-1$
(c) $(x+18)^2-40$	(d) $(x-10)^2-15$
(e) $-4(x+2)(x+3)$	(f) $(x+4)(x-6)$
(g) $(x-12)(-x+18)$	(h) $(20-x)(30-x)$



Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/640.

[F-IF.9] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions

The previous group of standards focuses on interpreting functions given by expressions, graphs, or tables. The Building Functions group focuses on building functions to model relationships, and building new functions from existing functions.

Note: Inverse of a function and composition of a function with its inverse are among the plus standards. The following discussion describes in detail what is required for students to grasp these securely. Because of the subtleties and pitfalls involved, it is strongly recommended that these topics be included only in optional courses.

Build a function that models a relationship between two quantities This cluster of standards is very closely related to the algebra standard on writing equations in two variables.[A-CED.2] Indeed, that algebra standard might well be met by a curriculum in the same unit as this cluster. Although students will eventually study various families of functions, it is useful for them to have experiences of building functions from scratch, without the aid of a host of special recipes, by grappling with a concrete context for clues.[F-BF.1a] For example, in "Lake Algae" in the margin, a solution for part (a) might involve noting that if the lake is completely covered with algae on June 30, then half of its surface will be covered on June 29 because the area covered doubles each day. This might be expressed in a table:

date	29	30
percent covered	50	100

Finding a solution for part (b) might start from the table above. Repeatedly using the information that the algae doubles each day: one divides the amount for June 29 by 2, then divides the amount for June 28 by 2, then divides the amount for June 27 by 2. This repeated reasoning (MP.8) might be suggested by the table:

date	26	27	28	29	30
percent covered	$\frac{1}{16}\cdot 100$	$\frac{1}{8}\cdot 100$	$\frac{1}{4}\cdot 100$	$\frac{1}{2}\cdot 100$	$1\cdot 100$

Some students might express the action of repeatedly dividing by 2 by writing the table entries for surface area as a product of 100 and a power of $\frac{1}{2}$ or 2, making use of structure (MP.7) by using an exponential expression. Or they might express this action with a recursively defined function, e.g., if t is a number between 2 and 30, and $f(t)$ gives the amount of surface covered on June t, then $f(t-1) = \frac{1}{2}f(t)$.

The Algebra Progression discusses the difference between a function and an expression. Not all functions are given by expressions, and in many situations it is natural to use a function defined recursively. Calculating mortgage payment and drug dosages are typical cases where recursively defined functions are useful (see "Drug Dosage" in the margin).

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-BF.1a Write a function that describes a relationship between two quantities.

a Determine an explicit expression, a recursive process, or steps for calculation from a context.

Lake Algae

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

(a) When will the lake be covered half-way?

(b) On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

(c) On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

(d) Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/533.

Drug Dosage

A student strained her knee in an intramural volleyball game, and her doctor has prescribed an anti-inflammatory drug to reduce the swelling. She is to take two 220-milligram tablets every 8 hours for 10 days. Her kidneys filter 60% of this drug from her body every 8 hours. How much of the drug is in her system after 24 hours?

Task from *High School Mathematics at Work: Essays and Examples for the Education of All Students*, 1998, National Academies Press. For discussion of the task, see http://www.nap.edu/openbook/0309063531/html/80.html.

Modeling contexts also provide a natural place for students to start building functions with simpler functions as components.[F-BF.1bc] Situations of cooling or heating involve functions which approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70° Fahrenheit and a cup of tea is made with boiling water at a temperature of 212° Fahrenheit, a student can express the function describing the temperature as a function of time using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, leading to a function of the form

$$T(t) = 70 + 142e^{-kt}.$$

Students might determine the constant k experimentally.

In contexts where change occurs at discrete intervals (such as payments of interest on a bank balance) or where the input variable is a whole number (for example the number of a pattern in a sequence of patterns), the functions chosen will be sequences. In preparation for the deeper study of linear and exponential functions, students can study arithmetic sequences (which are linear functions) and geometric sequences (which are exponential functions).[F-BF.2] This is a good point at which to start making the distinction between additive and multiplicative changes.

Build new functions from existing functions With a basis of experiences in building specific functions from scratch, students start to develop a notion of naturally occurring families of functions that deserve particular attention. It is possible to harden the curriculum too soon around these families, before students have enough experience to get a feel for the effects of different parameters. Students can start getting that feel by playing around with the effect on the graph of simple algebraic transformations of the input and output variables.[F-BF.3] Quadratic and absolute value functions are good contexts for getting a sense of the effects of many of these transformations, but eventually students need to understand these ideas abstractly and be able to talk about them for any function f.

Students may find the effect of adding a constant to the input variable to be counterintuitive, because the effect on the graph appears to be the opposite to the transformation on the variable, e.g., the graph of $y = f(x + 2)$ is a horizontal translation of the graph of $y = f(x)$ -2 units along the x-axis rather than in the opposite direction. In part (b) of “Transforming Functions” in the margin, asking students to talk through the positions of the points in terms of function values can help.•

The concepts of even and odd functions are useful for noticing symmetry. A function f is called an *even function* if $f(-x) = f(x)$ for all x in its domain and an *odd function* if $f(-x) = -f(x)$ for

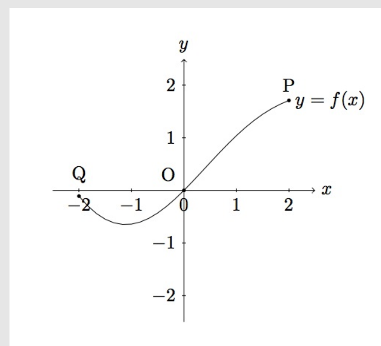
F-BF.1 Write a function that describes a relationship between two quantities.

- b Combine standard function types using arithmetic operations.
- c (+) Compose functions.

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Transforming Functions

The figure shows the graph of a function f whose domain is the interval $-2 \leqslant x \leqslant 2$.



(a) In (i)–(iii), sketch the graph of the given function and compare with the graph of f. Explain what you see.

- (i) $g(x) = f(x) + 2$
- (ii) $h(x) = -f(x)$
- (iii) $p(x) = f(x + 2)$

(b) The points labelled Q, O, P on the graph of f have coordinates

$$Q = (-2 - 0.509), \quad O = (0, -0.4), \quad P = (2, 1.309).$$

What are the coordinates of the points corresponding to P, O, Q on the graphs of g, h, and p?

Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/742.

F-BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

• The graphs of linear functions are especially complicated with respect to adding a constant to the input variable because its effect can be seen as one of many different translations. For example, the graph of $y = 2(x + 3)$ can be seen as a horizontal translation of the graph of $y = 2x$. But, thinking of it as $y = 2x+6$ it can also be seen as a vertical translation that moves the graph 6 units. And, it can also be seen as a translation in other directions, e.g., as suggested by $y = 2(x + 3 - c) + 2c$.

all x in its domain. To understand the names of these concepts, consider that polynomial functions are even exactly when all terms are of even degree and odd exactly when all terms are of odd degree. With some grounding in polynomial functions, students can reason that lots of functions are neither even nor odd.

Students can show from the definitions that the sum of two even functions is even and the sum of two odd functions is odd, and they can interpret these results graphically.

An interesting fact

Suppose f is a function with a domain of all real numbers. Define g and h as follows:

$$g(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad h(x) = \frac{f(x) - f(-x)}{2}$$

Then $f(x) = g(x) + h(x)$, g is even, and h is odd. (Students may use the definitions to verify these claims.) Thus, any function defined on the real numbers can be expressed as the sum of an even and an odd function.

When it comes to inverse functions,[F-BF.4a] the expectations are modest, requiring only that students solve equations of the form $f(x) = c$. The point is to provide an informal sense of determining the input when the output is known. Much of this work can be done with specific values of c. Eventually, some generality is warranted. For example, if $f(x) = 2x^3$, then solving $f(x) = c$ leads to $x = (c/2)^{1/3}$, which is the general formula for finding an input from a specific output, c, for this function, f.

F-BF.4a Find inverse functions.

a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

At this point, students need neither the notation nor the formal language of inverse functions, but only the idea of "going backwards" from output to input. This can be interpreted for a table and graph of the function under examination. Correspondences between equations giving specific values of the functions, table entries, and points on the graph can be noted (MP.1). And although not required in the standard, it is reasonable to include, for comparison, a few examples where the input cannot be uniquely determined from the output. For example, if $g(x) = x^2$, then $g(x) = 5$ has two solutions, $x = \pm\sqrt{5}$.

For some advanced mathematics courses, students will need a formal sense of inverse functions, which requires careful development. For example, as students begin formal study, they can easily believe that "inverse functions" are a new family of functions, similar to linear functions and exponential functions. To help students develop the instinct that "inverse" is a relationship between two functions, the recurring questions should be "What is the inverse of this function?" and "Does this function have an inverse?" The focus should be on "inverses of functions" rather than a new type of function.

A joke

Teacher: Are these two functions inverses?
Student: Um, the first one is and the second one isn't.

What does this student misunderstand about inverse functions?

Discussions of the language and notation for inverse functions can help to provide students a sense of what the adjective "inverse" means and mention that a function which has an inverse is known as an "invertible function."

The function $\mathcal{I}(x) = x$ is sometimes called the identity function because it assigns each number to itself. It behaves with respect to composition of functions the way the multiplicative identity, 1, behaves with multiplication of real numbers and the way that the identity matrix behaves with matrix multiplication. If f is any function (defined on the real numbers), this analogy can be expressed symbolically as $f \circ \mathcal{I} = f = \mathcal{I} \circ f$, and it can be verified as follows:

$$f \circ \mathcal{I}(x) = f(\mathcal{I}(x)) = f(x) \qquad \text{and} \qquad \mathcal{I} \circ f(x) = \mathcal{I}(f(x)) = f(x)$$

Suppose f denotes a function with an inverse whose domain is the real numbers and a is a nonzero real number (which thus has a multiplicative inverse), and B is an invertible matrix. The following table compares the concept of inverse function with the concepts of multiplicative inverse and inverse matrix:

Equation	Interpretation
$f^{-1} \circ f = \mathcal{I} = f \circ f^{-1}$	The composition of f^{-1} with f is the identity function
$a^{-1} \cdot a = 1 = a \cdot a^{-1}$	The product of a^{-1} and a is the multiplicative identity
$B^{-1} \cdot B = \mathcal{I} = B \cdot B^{-1}$	The product of B^{-1} and B is the identity matrix

In other words, where a^{-1} means the inverse of a with respect to multiplication, f^{-1} means the inverse of f with respect to function composition. Thus, when students interpret the notation $f^{-1}(x)$ incorrectly to mean $1/f(x)$, the guidance they need is that the meaning of the "exponent" in f^{-1} is about function composition, not about multiplication.

A note on notation

In the expression $\sin^2 x$, the superscript denotes exponentiation. In $\sin^{-1} x$, the superscript denotes inverse with respect to composition of functions rather than with respect to multiplication. Despite the similar look, these superscripts act in different ways. The 2 acts as an exponent but the -1 does not. Both notations, however, allow the expression to be written without the parentheses that would be needed otherwise.

Another convention that allows parentheses to be omitted is the use of $\sin ax$ rather than $\sin(ax)$. Thus, some expressions built from trigonometric functions may written in ways that look quite different to students, but differ only in the use or omission of parentheses.

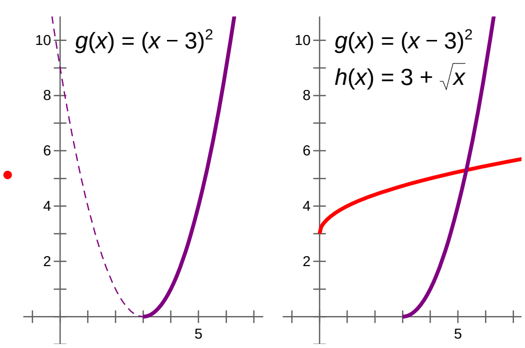
Students do not need to develop the abstract sense of identity and inverse detailed in this table. Nonetheless, these perspectives can inform the language and conversation in the classroom as students verify by composition (in both directions) that given functions are inverses of each other.[F-BF.4b]. Furthermore, students can continue to refine their informal "going backwards" notions, as they consider inverses of functions given by graphs or tables.[F-BF.4c] In this work, students can gain a sense that "going backwards" interchanges the input and output and therefore the stereotypical roles of the letters x and y. And they can reason why the graph of $y = f^{-1}(x)$ will be the reflection across the line $y = x$ of the graph of $y = f(x)$.

F-BF.4b(+) Verify by composition that one function is the inverse of another.

F-BF.4c(+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Suppose $g(x) = (x-3)^2$. From the graph, it can be seen that $g(x) = c$ will have two solutions for any $c > 0$. (This draws on the understanding that solutions of $g(x) = c$ are x-coordinates of points that lie on both the graphs of g and $y = c$.) Thus, to create an invertible function,[F-BF.4d] we must restrict the domain of g so that every range value corresponds to exactly one domain value. One possibility is to restrict the domain of g to $x \geqslant 3$, as illustrated by the solid purple curve in the graph on the left.

F-BF.4d(+) Produce an invertible function from a non-invertible function by restricting the domain.



When solving $(x-3)^2 = c$, we get $x = 3 \pm \sqrt{c}$, illustrating that positive values of c will yield two solutions x for the unrestricted function. With the restriction, $3 - \sqrt{c}$ is not in the domain. Thus, $x = 3 + \sqrt{c}$, which corresponds to choosing the solid curve and ignoring the dotted portion. The inverse function, then, is $h(c) = 3 + \sqrt{c}$, for $c \geqslant 0$.

We check that h is the inverse of (restricted) g as follows:

$$g(h(x)) = g\left(3 + \sqrt{x}\right) = \left((3 + \sqrt{x}) - 3\right)^2 = (\sqrt{x})^2 = x, \quad x \geqslant 0$$

$$h(g(x)) = h\left((x-3)^2\right) = 3 + \sqrt{(x-3)^2} = 3 + (x-3) = x, \quad x \geqslant 3.$$

The first verification requires that $x \geqslant 0$ so that x is in the domain of h. The second verification requires that $x \geqslant 3$ so that x is in the domain of (restricted) g. This allows $\sqrt{(x-3)^2}$ to be written without the square root symbol as $(x-3)$.• The rightmost graph shows the graph of h. Students can draw on their work with transformations in Grades 7 and 8,[8.G.3] possibly augmented by plotting points such as (0,3) and (3,0), to perceive the graph of h as the reflection of the graph of g across the line $y = x$.

• In general, $\sqrt{(x-3)^2} = |x-3|$. If x were restricted to the dotted portion of the graph (i.e., $x \leqslant 3$), the corresponding expression could have been written as $-(x-3)$ or $3-x$.

8.G.3 Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

Linear and Exponential Models

Construct and compare linear and exponential models and solve problems Distinguishing between situations that can be modeled with linear functions and with exponential functions[F-LE.1a] turns on understanding their rates of growth and looking for indications of these types of growth rates (MP.7). One indicator of these growth rates is differences over equal intervals, given, for example, in a table of values drawn from the situation—with the understanding that such a table may only approximate the situation (MP.4).

F-LE.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

To prove that a linear function grows by equal differences over equal intervals,[F-LE.1b] students draw on the understanding developed in Grade 8 that the ratio of the rise and run for any two distinct points on a line is the same (see the Expressions and Equations Progression) and recast it in terms of function inputs and outputs. An interval can be seen as determining two points on the line whose inputs (x-coordinates) occur at the boundaries of the intervals. The equal intervals can be seen as the runs for two pairs of points. Because these runs have equal length and the ratio of rise to run is the same for any pair of distinct points, the differences of the corresponding outputs (the rises) are the same. These differences are the growth of the function over each interval.

F-LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

In the process of this proof, students note the correspondence between rise and run on a graph and symbolic expressions for differences of inputs and outputs (MP.1). Using such expressions has the advantage that the analogous proof showing that exponential functions grow by equal factors over equal intervals begins in an analogous way with expressions for differences of inputs and outputs.

The process of going from linear or exponential functions to tables can go in the opposite direction. Given sufficient information, e.g., a table of values together with information about the type of relationship represented,[F-LE.4] students construct the appropriate function. For example, students might be given the information that the table below shows inputs and outputs of an exponential function, and asked to write an expression for the function.

F-LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Input	Output
0	5
8	33

For most students, the logarithm of x is merely shorthand for a number that is the solution of an exponential equation in x.[F-LE.4]

Students in advanced mathematics courses such as calculus, however, need to understand logarithms as functions—and as inverses of exponential functions.[F-BF.5] They should be able to explain identities such as $\log_b(b^x) = x$ and $b^{\log_b x} = x$ as well as the laws of logarithms, such as $\log(ab) = \log a + \log b$. In doing so, students can think of the logarithms as unknown exponents in expressions with base 10 (e.g. $\log a$ answers the question "Ten to the what

F-LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

+ equals a?") and use the properties of exponents,[N-RN.1] building on
+ the understanding of exponents that began in Grade 8.[8.EE.1]

Interpret expressions for functions in terms of the situation they model Students may build a function to model a situation, using parameters from that situation. In these cases, interpreting expressions for a linear or exponential function in terms of the situation it models[F-LE.5] is often just a matter of remembering how the function was constructed. However, interpreting expressions may be less straightforward for students when they are given an algebraic expression for a function and a description of what the function is intended to model.

For example, in doing the task "Illegal Fish" in the margin, students may need to rely on their understanding of a function as determining an output for a given input to answer the question "Find b if you know the lake contains 33 fish after eight weeks."

See the linear and exponential model section of the Modeling Progression for an example of an interpretation of the intersection of a linear and an exponential function in terms of the situation that is being modeled.

N-RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

Illegal Fish

A fisherman illegally introduces some fish into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where x is the time in weeks following the introduction and b is a positive unknown base.

(a) Exactly how many fish did the fisherman release into the lake?

(b) Find b if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

(c) Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

Task from Illustrative Mathematics. For solutions and discussion, see illustrativemathematics.org/illustrations/579.

Trigonometric Functions

Students begin their study of trigonometry with right triangles.[G-SRT.6] Right triangle trigonometry is concerned with ratios of sides of right triangles, allowing functions of angle measures to be defined in terms of these ratios.• This limits the angles considered to those between $0°$ and $90°$. This section briefly outlines some considerations involved in extending the domains of the trigonometric functions within the real numbers.

Traditionally, trigonometry includes six functions (sine, cosine, tangent, cotangent, secant, cosecant). Because the second three may be expressed as reciprocals of the first three, this progression discusses only the first three.

G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

• Traditionally, trigonometry concerns "ratios." Note, however, that according to the usage of the Ratio and Proportional Relationships Progression, that these would be called the "value of the ratio." In high school, students' understanding of ratio may now be sophisticated enough to allow the traditional "ratio" to be used for "value of the ratio" in the traditional manner. Likewise, angles are carefully distinguished from their measurements when students are learning about measuring angles in Grades 4 and 5. In high school, students' understanding of angle measure may now allow angles to be referred to by their measures.

Extend the domain of trigonometric functions using the unit circle After study of trigonometric ratios in right triangles, students expand the types of angles considered. Students learn, by similarity, that the radian measure of an angle can be defined as the quotient of arc length to radius.[G-C.5] As a quotient of two lengths, therefore, radian measure is "dimensionless." That is why the "unit" is often omitted when measuring angles in radians.

G-C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

In calculus, the benefits of radian measure become plentiful, leading, for example, to simple formulas for derivatives and integrals of trigonometric functions. Before calculus, there are two key benefits of using radians rather than degrees:

- arclength is simply $r\theta$, and
- $\sin\theta \approx \theta$ for small θ.

Steps to extending the domain of trigonometric functions and introduction of radian measurement may include:

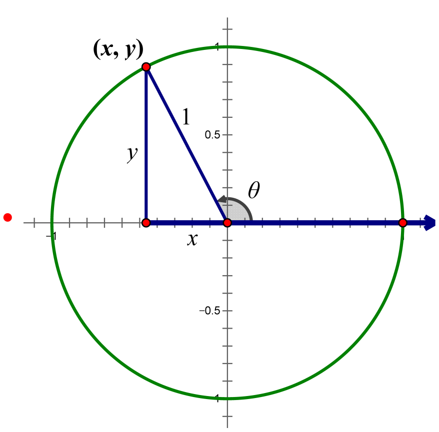
- Extending consideration of trigonometric ratios from right triangles to obtuse triangles. This may occur in the context of solving problems about geometric figures.[G-SRT.5][G-SRT.9] See the Geometry Progression.
- Associating the degree measure of an angle with the length of the arc it subtends on the unit circle,[F-TF.1] as described below.

G-SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-SRT.9(+) Derive the formula $A = 1/2ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

With the help of a diagram, students mark the intended angle, θ, measured counterclockwise from the positive ray of the x-axis.• They identify the coordinates x and y; draw a reference triangle; and then use their knowledge of right triangle trigonometry. In particular, $\sin\theta = y/1 = y$, $\cos\theta = x/1 = x$, and $\tan\theta = y/x$. (Note the simplicity afforded by using a circle of radius 1.) This way, students can compute values of any of the trigonometric functions, being careful to note the signs of x and y. In the figure as drawn in the second quadrant, for example, x is negative and y is positive,



• Note that this convention for measurement is consistent with conventions for measuring angles with protractors that students learned in Grade 4. The protractor is placed so that the initial side of the angle lies on the $0°$-mark. For the angles of positive measure (such as the angles considered in Grade 4), the terminal side of the angle is located by a clockwise rotation. See the Geometric Measurement Progression.

which implies that $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are both negative.

The next step is sometimes called "unwrapping the unit circle." On a fresh set of axes, the angle θ is plotted along the horizontal axis and one of the trigonometric functions is plotted along the vertical axis. Dynamic presentations with shadows can help considerably, and the point should be that students notice the periodicity of the functions, caused by the repeated rotation about the origin, regularly reflecting on the grounding in right triangle trigonometry.

With the help of the special right triangles, $30°$-$60°$-$90°$ and $45°$-$45°$-$90°$, for which the quotients of sides can be computed using the Pythagorean Theorem,[8.G.7] the values of the trigonometric functions can be computed for the angles $\pi/3$, $\pi/4$, and $\pi/6$ as well as their multiples.[F-TF.3] For advanced mathematics, students need to develop fluency with the trigonometric functions of these special angles to support fluency with the "unwrapping of the unit circle" to create and graph the trigonometric functions.

Building on their understanding of geometric transformations,[G-CO.7] either directly or via the side-angle-side congruence criterion,[G-CO.8] students see that, compared to the reference triangle with angle θ, an angle of $-\theta$ will produce a congruent reference triangle that is its reflection across the x-axis. They can then reason that $\sin(-\theta) = -y = -\sin(\theta)$, so sine is an odd function. Similarly, $\cos(-\theta) = x = \cos(\theta)$, so cosine is an even function.[F-TF.4] Some additional work is required to verify that these relationships hold for values of θ outside the first quadrant.

The same sorts of pictures can be used to argue that the trigonometric functions are periodic. For example, for any integer n, $\sin(\theta + 2n\pi) = \sin(\theta)$ because angles that differ by a multiple of 2π have the same terminal side and thus the same coordinates x and y.

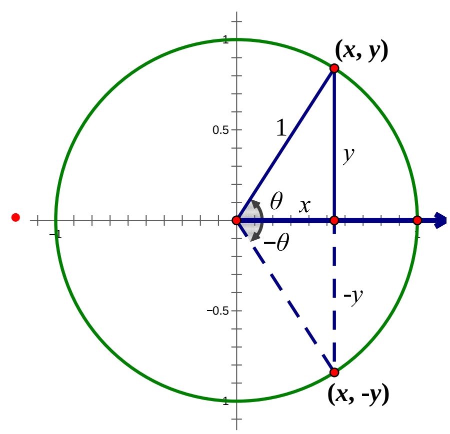
Model periodic phenomena with trigonometric functions Now that students are equipped with trigonometric functions, they can model some periodic phenomena that occur in the real world. For students who do not continue into advanced mathematics, this is the culmination of their study of trigonometric functions.

The tangent function is not often useful for modeling periodic phenomena because $\tan x$ is undefined for $x = \frac{\pi}{2} + k\pi$, where k is an integer. Because the graphs of sine and cosine have the same shape (each is a horizontal translation of the other), either suffices to model simple periodic phenomena.[F-TF.5] A function is described as *sinusoidal* or is called a *sinusoid* if it has the same shape as the sine graph, i.e. has the form $f(t) = A + B\sin(Ct + D)$. Many real-world phenomena can be approximated by sinusoids, including sound waves, oscillation on a spring, the motion of a pendulum, tides, and phases of the moon. Some students will learn in college that sinusoids are used as building blocks to approximate any periodic waveform.

8.G.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

G-CO.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.



G-CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

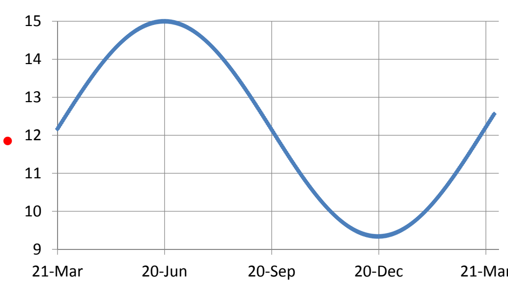
Because $\sin t$ oscillates between -1 and 1, $A + B\sin(Ct + D)$ will oscillate between $A - B$ and $A + B$. Thus, $y = A$ is the midline, and B is the amplitude of the sinusoid. Students can obtain the frequency of f: the period of $sint$ is 2π, so (knowing the effect of multiplying t by C) the period of $sinCt$ is $2\pi/C$, and the frequency is its reciprocal. When modeling, students need to have the sense that C affects the frequency and that C and D together produce a phase shift, but getting these correct might involve technological support, except in simple cases.

Frequency vs. period

For a sinusoid, the frequency is often measured in cycles per time unit, thus the period is often measured in time unit per cycle. For reasoning about a context, it is common to choose whichever is greater numerically.

For example, students might be asked to model the length of the day in Columbus, Ohio. Day length as a function of date is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference. So $A \approx 12.17$, and $B \approx 2.83$. With some support, students can determine that for the period to be 365 days (per cycle) (or the frequency to be $\frac{1}{365}$ cycles/day), $C = 2\pi/365$, and if day 0 corresponds to March 21, no phase shift is needed, so $D = 0$. Thus,

$$f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$$

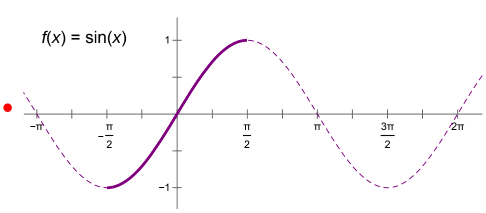


From the graph, students can see that the period is indeed 365 days, as desired, as it takes one year to complete the cycle. They can also see that two days are approximately 14 hours long, which is to say that $f(t) = 14$ has two solutions over a domain of one year, and they might use graphing or spreadsheet technology to determine that May 1 and August 10 are the closest such days. Students can also see that $f(t) = 9$ has no solutions, which makes sense because 9 hours, 20 minutes is the minimum length of day.

Students who take advanced mathematics will need additional fluency with transformations of trigonometric functions, including changes in frequency and phase shifts.

Based on plenty of experience solving equations of the form $f(t) = c$ graphically, students of advanced mathematics should be able to see that such equations will have an infinite number of solutions when f is a trigonometric function. Furthermore, they should have had experience of restricting the domain of a function so that it has an inverse. For trigonometric functions, a common approach to restricting the domain is to choose an interval on which the function is always increasing or always decreasing.[F-TF.6] The obvious choice for $\sin(x)$ is the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, shown as the solid part of the graph. This yields a function $\theta = \sin^{-1}(x)$ with domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.



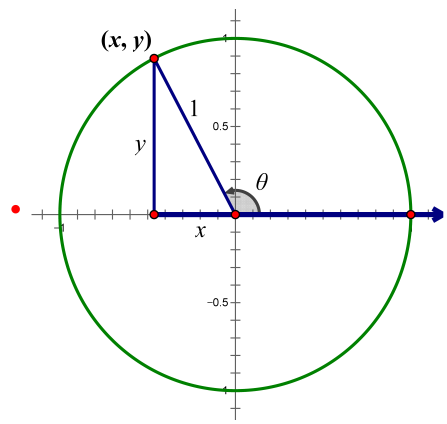
Inverses of trigonometric functions can be used in solving equations in modeling contexts.[F-TF.7] For example, in the length of day context, students can use inverse trig functions to determine days with particular lengths. This amounts to solving $f(t) = d$ for t, which

F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

yields

$$t = \frac{365}{2\pi} \sin^{-1}\left(\frac{d - 12.17}{2.83}\right)$$

Using $d = 14$ and a calculator (in radian mode), they can compute that $t \approx 40.85$, which is closest to May 1. Finding the other solution is a bit of a challenge, but the graph indicates that it should occur just as many days before midyear (day 182.5) as May 1 occurs after day 0. So the other solution is $t \approx 182.5 - 40.85 = 141.65$, which is closest to August 10.



Prove and apply trigonometric identities For the cases illustrated by the diagram (in which the terminal side of angle θ does not lie on an axis) and the definitions of $\sin\theta$ and $\cos\theta$, students can reason that, in any quadrant, the lengths of the legs of the reference triangle are $|x|$ and $|y|$. It then follows from the Pythagorean Theorem that $|x|^2 + |y|^2 = 1$. Because $|a|^2 = a^2$ for any real number a, this equation can be written $x^2 + y^2 = 1$. Because $x = \cos\theta$ and $y = \sin\theta$, the equation can be written as $\sin^2(\theta) + \cos^2(\theta) = 1$. When the terminal side of angle θ does lie on an axis, then one of x or y is 0 and the other is 1 or -1 and the equation still holds. This argument proves what is known as the Pythagorean identity[F-TF.8] because it is essentially a restatement of the Pythagorean Theorem for a right triangle of hypotenuse 1.

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

With this identity and the value of one of the trigonometric functions for a given angle, students can find the values of the other functions for that angle, as long as they know the quadrant in which the angle lies. For example, if $\sin(\theta) = 0.6$ and θ lies in the second quadrant, then $\cos^2(\theta) = 1 - 0.6^2 = 0.64$, so $\cos(\theta) = \pm\sqrt{0.64} = \pm 0.8$. Because cosine is negative in the second quadrant, it follows that $\cos(\theta) = 0.8$, and therefore $\tan(\theta) = \sin(\theta)/\cos(\theta) = 0.6/(-0.8) = -0.75$.

Students in advanced mathematics courses can prove and use other trigonometric identities, including the addition and subtraction formulas.[F-TF.9] If students have already represented complex numbers on the complex plane[N-CN.4] and developed the geometric interpretation of their multiplication,[N-CN.5] then the the product $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ can be used in deriving the addition formulas for cosine and sine. Subtraction and double angle formulas can follow from these.

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

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For updates and more information about the Progressions, see **http://ime.math.arizona.edu/progressions**.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: **http://commoncoretools.me**.

High School, Algebra*

Overview

Two Grades 6–8 domains are important in preparing students for Algebra in high school. The Number System prepares students to see all numbers as part of a unified system, and become fluent in finding and using the properties of operations to find the values of numerical expressions that include those numbers. The standards of the Expressions and Equations domain ask students to extend their use of these properties to linear equations and expressions with letters. These extend uses of the properties of operations in earlier grades: in Grades 3–5 Number and Operations—Fractions, in K–5 Operations and Algebraic Thinking, and K–5 Number and Operations in Base Ten.

The Algebra category in high school is very closely allied with the Functions category:

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression at a given input is the act of producing the function's output at that input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. For example, this is the case if the equation is in the form $y =$ (expression in x) or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent they define the same function.

*The study of algebra occupies a large part of a student's high school career, and this document does not treat in detail all of the material studied. Rather it gives some general guidance about ways to treat the material and ways to tie it together. It notes key connections among standards, points out cognitive difficulties and pedagogical solutions, and gives more detail on particularly knotty areas of the mathematics.

The high school standards specify the mathematics that all students should study in order to be college and career ready. Additional material corresponding to (+) standards, mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics, is indicated by plus signs in the left margin.

- The solutions to an equation in one variable can be understood as the input values which yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing the functions defined by each side and finding the points where the graphs intersect.

Because of these connections, some curricula take a functions-based approach to teaching algebra, in which functions are introduced early and used as a unifying theme for algebra. Other approaches introduce functions later, after extensive work with expressions and equations. The separation of algebra and functions in the Standards is not intended to indicate a preference between these two approaches. It is, however, intended to specify the difference as mathematical concepts between expressions and equations on the one hand and functions on the other. Students often enter college-level mathematics courses apparently conflating all three of these. For example, when asked to factor a quadratic expression a student might instead find the solutions of the corresponding quadratic equation. Or another student might attempt to simplify the expression $\frac{\sin x}{x}$ by cancelling the x's.

The algebra standards are fertile ground for the Standards for Mathematical Practice. Two in particular that stand out are MP.7, "Look for and make use of structure" and MP.8 "Look for and express regularity in repeated reasoning." Students are expected to see how the structure of an algebraic expression reveals properties of the function it defines. They are expected to move from repeated reasoning with pairs of points on a line to writing equations in various forms for the line, rather than memorizing all those forms separately. In this way the Algebra standards provide focus in a way different from the K–8 standards. Rather than focusing on a few topics, students in high school focus on a few seed ideas that lead to many different techniques.

Seeing Structure in Expressions

Students have been seeing expressions since Kindergarten, starting with arithmetic expressions in Grades K–5 and moving on to algebraic expressions in Grades 6–8. The middle grades standards in Expression and Equations build a ramp from arithmetic expressions in elementary school to more sophisticated work with algebraic expressions in high school. As the complexity of expressions increases, students continue to see them as being built out of basic operations: they see expressions as sums of terms and products of factors.[A-SSE.1a]

For example, in "Animal Populations" in the margin, students compare $P + Q$ and $2P$ by seeing $2P$ as $P + P$. They distinguish between $(Q - P)/2$ and $Q - P/2$ by seeing the first as a quotient where the numerator is a difference and the second as a difference where the second term is a quotient. This last example also illustrates how students are able to see complicated expressions as built up out of simpler ones.[A-SSE.1b] As another example, students can see the expression $5 + (x - 1)^2$ as a sum of a constant and a square; and then see that inside the square term is the expression $x - 1$. The first way of seeing tells them that it is always greater than or equal to 5, since a square is always greater than or equal to 0; the second way of seeing tells them that the square term is zero when $x = 1$. Putting these together they can see that this expression attains its minimum value, 5, when $x = 1$. The margin lists other tasks from the Illustrative Mathematics project (illustrativemathematics.org) for A-SSE.1.

In elementary grades, the repertoire of operations for building expressions is limited to the operations of arithmetic: addition, subtraction, multiplication and division. Later, it is augmented by exponentiation, first with whole numbers in Grades 5 and 6, then with integers in Grade 8. By the time they finish high school, students have expanded that repertoire to include radicals and trigonometric expressions, along with a wider variety of exponential expressions.

For example, students in physics classes might be expected to see the expression

$$L_0\sqrt{1 - \frac{v^2}{c^2}},$$

which arises in the theory of special relativity, as the product of the constant L_0 and a term that is 1 when $v = 0$ and 0 when $v = c$—and furthermore, they might be expected to see it without having to go through a laborious process of written or electronic evaluation. This involves combining the large-scale structure of the expression—a product of L_0 and another term—with the structure of internal components such as $\frac{v^2}{c^2}$.

Seeing structure in expressions entails a dynamic view of an algebraic expression, in which potential rearrangements and manipulations are ever present.[A-SSE.2] An important skill for college

Animal Populations

Suppose P and Q give the sizes of two different animal populations, where $Q > P$. In 1–4, say which of the given pair of expressions is larger. Briefly explain your reasoning in terms of the two populations.

1. $P + Q$ and $2P$
2. $\frac{P}{P+Q}$ and $\frac{P+Q}{2}$
3. $(Q - P)/2$ and $Q - P/2$
4. $P + 50t$ and $Q + 50t$

Task from Illustrative Mathematics. For solutions and discussion, see http://www.illustrativemathematics.org/illustrations/436.

A-SSE.1a Interpret expressions that represent a quantity in terms of its context.

a Interpret parts of an expression, such as terms, factors, and coefficients.

A-SSE.1b Interpret expressions that represent a quantity in terms of its context.

b Interpret complicated expressions by viewing one or more of their parts as a single entity.

Illustrations of interpreting the structure of expressions

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Delivery Trucks
- Kitchen Floor Tiles
- Increasing or Decreasing? Variation 1
- Mixing Candies
- Mixing Fertilizer
- Quadrupling Leads to Halving
- The Bank Account
- The Physics Professor
- Throwing Horseshoes
- Animal Populations
- Equivalent Expressions
- Sum of Even and Odd

A-SSE.2 Use the structure of an expression to identify ways to rewrite it.

readiness is the ability to try possible manipulations mentally without having to carry them out, and to see which ones might be fruitful and which not. For example, a student who can see

$$\frac{(2n+1)n(n+1)}{6}$$

as a polynomial in n with leading coefficient $\frac{1}{3}n^3$ has an advantage when it comes to calculus; a student who can mentally see the equivalence

$$\frac{R_1R_2}{R_1+R_2} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$$

without a laborious pencil and paper calculation is better equipped for a course in electrical engineering.

The Standards avoid talking about simplification, because it is often not clear what the simplest form of an expression is, and even in cases where that is clear, it is not obvious that the simplest form is desirable for a given purpose. The Standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as illustrated in the margin.[A-SSE.3]

For example, there are three commonly used forms for a quadratic expression:

- Standard form, e.g., $x^2 - 2x - 3$
- Factored form, e.g., $(x+1)(x-3)$
- Vertex form (a square plus or minus a constant), e.g. $(x-1)^2-4$.

Rather than memorize the names of these forms, students need to gain experience with them and their different uses. The traditional emphasis on simplification as an automatic procedure might lead students to automatically convert the second two forms to the first, rather than convert an expression to a form that is useful in a given context.[A-SSE.3ab] This can lead to time-consuming detours in algebraic work, such as solving $(x+1)(x-3) = 0$ by first expanding and then applying the quadratic formula.

The introduction of rational exponents and systematic practice with the properties of exponents in high school widen the field of operations for manipulating expressions.[A-SSE.3c] For example, students in later algebra courses who study exponential functions see

$$P(1+\frac{r}{12})^{12n} \quad \text{as} \quad P\left((1+\frac{r}{12})^{12}\right)^n$$

in order to understand formulas for compound interest.

Much of the ability to see and use structure in transforming expressions comes from learning to recognize certain fundamental situations that afford particular techniques. One such technique is internal cancellation, as in the expansion

$$(a-b)(a+b) = a^2 - b^2.$$

Which form is "simpler"?

After a container of ice cream has been sitting in a room for t minutes, its temperature in degrees Fahrenheit is

$$a - b \cdot 2^{-t} + b,$$

where a and b are positive constants.

Write this expression in a form that

1. Shows that the temperature is always less than $a+b$.
2. Shows that the temperature is never less than a.

Commentary The form $a + b - b \cdot 2^{-t}$ for the temperature shows that it is $a+b$ minus a positive number, so always less than $a+b$. On the other hand, the form $a + b(1-2^{-t})$ reveals that the temperature is never less than a, because it is a plus a positive number.

"Ice Cream" task from Illustrative Mathematics. For solutions and discussion, see http://www.illustrativemathematics.org/illustrations/551.

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a Factor a quadratic expression to reveal the zeros of the function it defines.

b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c Use the properties of exponents to transform expressions for exponential functions.

Illustrations of writing expressions in equivalent forms

The following tasks can be found by going to http://illustrativemathematics.org/illustrations/ and searching for A-SSE:

- Ice Cream
- Increasing or Decreasing? Variation 2
- Profit of a Company
- Seeing Dots

An impressive example of this is

$$(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)=x^n-1,$$

in which all the terms cancel except the end terms. This identity is the foundation for the formula for the sum of a finite geometric series.[A-SSE.4]

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Arithmetic with Polynomials and Rational Expressions

The development of polynomials and rational expressions in high school parallels the development of numbers in elementary and middle grades. In elementary school, students might initially see expressions for the same numbers $8+3$ and 11, or $\frac{3}{4}$ and 0.75, as referring to different entities: $8+3$ might be seen as describing a calculation and 11 is its answer; $\frac{3}{4}$ is a fraction and 0.75 is a decimal. They come to understand that these different expressions are different names for the same numbers, that properties of operations allow numbers to be written in different but equivalent forms, and that all of these numbers can be represented as points on the number line. In middle grades, they come to see numbers as forming a unified system, the number system, still represented by points on the number line. The whole numbers expand to the integers—with extensions of addition, subtraction, multiplication, and division, and their properties. Fractions expand to the rational numbers—and the four operations and their properties are extended.

A similar evolution takes place in algebra. At first algebraic expressions are simply numbers in which one or more letters are used to stand for a number which is either unspecified or unknown. Students learn to use the properties of operations to write expressions in different but equivalent forms. At some point they see equivalent expressions, particularly polynomial and rational expressions, as naming some underlying thing.[A-APR.1] There are at least two ways this can go. If the function concept is developed before or concurrently with the study of polynomials, then a polynomial can be identified with the function it defines. In this way x^2-2x-3, $(x+1)(x-3)$, and $(x-1)^2-4$ are all the same polynomial because they all define the same function. Another approach is to think of polynomials as elements of a formal number system, in which you introduce the "number" x and see what numbers you can write down with it. In this approach, x^2-2x-3, $(x+1)(x-3)$, and $(x-1)^2-4$ are all the same polynomial because the properties of operations allow each to be transformed into the others. Each approach has its advantages and disadvantages; the former approach is more common. Whichever is chosen and whether or not the choice is explicitly stated, a curricular implementation should nonetheless be constructed to be consistent with the choice that has been made.

A-APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Either way, polynomials and rational expressions come to form a system in which they can be added, subtracted, multiplied and divided.[A-APR.7] Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers.

A-APR.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers.[A-APR.4] For example, students can explore the sequence of squares

A-APR.4 Prove polynomial identities and use them to describe numerical relationships.

$$1, 4, 9, 16, 25, 36, \ldots$$

and notice that the differences between them—3, 5, 7, 9, 11—are consecutive odd integers. This mystery is explained by the polynomial identity

$$(n+1)^2 - n^2 = 2n + 1.$$

A more complex identity,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2,$$

allows students to generate Pythagorean triples. For example, taking $x = 2$ and $y = 1$ in this identity yields $5^2 = 3^2 + 4^2$.

A particularly important polynomial identity, treated in advanced courses, is the Binomial Theorem[A-APR.5]

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \binom{n}{3} x^{n-3} y^3 + \cdots + y^n,$$

for a positive integer n. The binomial coefficients can be obtained using Pascal's triangle

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

in which each entry is the sum of the two above. Understanding why this rule follows algebraically from

$$(x+y)(x+y)^{n-1} = (x+y)^n$$

is excellent exercise in abstract reasoning (MP.2) and in expressing regularity in repeated reasoning (MP.8).

A-APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Polynomials as functions Viewing polynomials as functions leads to explorations of a different nature. Polynomial functions are, on the one hand, very elementary, in that, unlike trigonometric and exponential functions, they are built up out of the basic operations of arithmetic. On the other hand, they turn out to be amazingly flexible, and can be used to approximate more advanced functions such as trigonometric and exponential functions. Experience with constructing polynomial functions satisfying given conditions is useful preparation not only for calculus (where students learn more about approximating functions), but for understanding the mathematics behind curve-fitting methods used in applications to statistics and computer graphics.

A simple step in this direction is to construct polynomial functions with specified zeros.[A-APR.3] This is the first step in a progression which can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

A-APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Polynomials as analogues of integers The analogy between polynomials and integers carries over to the idea of division with remainder. Just as in Grade 4 students find quotients and remainders of integers,[4.NBT.6] in high school they find quotients and remainders of polynomials.[A-APR.6] The method of polynomial long division is analogous to, and simpler than, the method of integer long division.

A particularly important application of polynomial division is the case where a polynomial $p(x)$ is divided by a linear factor of the form $x - a$, for a real number a. In this case the remainder is the value $p(a)$ of the polynomial at $x = a$.[A-APR.2] It is a pity to see this topic reduced to "synthetic division," which reduced the method to a matter of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique.

A consequence of the Remainder Theorem is to establish the equivalence between linear factors and zeros that is the basis of much work with polynomials in high school: the fact that $p(a) = 0$ if and only if $x - a$ is a factor of $p(x)$. It is easy to see if $x - a$ is a factor then $p(a) = 0$. But the Remainder Theorem tells us that we can write

$$p(x) = (x - a)q(x) + p(a) \quad \text{for some polynomial } q(x).$$

In particular, if $p(a) = 0$ then $p(x) = (x - a)q(x)$, so $x - a$ is a factor of $p(x)$.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

A-APR.6 Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

A-APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Creating Equations

Students have been seeing and writing equations since elementary grades,[K.OA.1, 1.OA.1] with mostly linear equations in middle grades. At first glance it might seem that the progression from middle grades to high school is fairly straightforward: the repertoire of functions that is acquired during high school allows students to create more complex equations, including equations arising from linear and quadratic expressions, and simple rational and exponential expressions;[A-CED.1] students are no longer limited largely to linear equations in modeling relationships between quantities with equations in two variables;[A-CED.2] and students start to work with inequalities and systems of equations.[A-CED.3]

Two developments in high school complicate this picture. First, students in high school start using parameters in their equations, to represent whole classes of equations[F-LE.5] or to represent situations where the equation is to be adjusted to fit data.•

Second, modeling becomes a major objective in high school. Two of the standards just cited refer to "solving problems" and "interpreting solutions in a modeling context." And all the standards in the Creating Equations group carry a modeling star, denoting their connection with the Modeling category in high school. This connotes not only an increase in the complexity of the equations studied, but an upgrade of the student's ability in every part of the modeling cycle, shown in the margin.

Variables, parameters, and constants Confusion about these terms plagues high school algebra. Here we try to set some rules for using them. These rules are not purely mathematical; indeed, from a strictly mathematical point of view there is no need for them at all. However, users of equations, by referring to letters as "variables," "parameters," or "constants," can indicate how they intend to use the equations. This usage can be helpful if it is consistent.

In elementary and middle grades, students solve problems with an unknown quantity, might use a symbol to stand for that quantity, and might call the symbol an unknown.[1.OA.2] In Grade 6, students begin to use variables systematically.[6.EE.6] They work with equations in one variable, such as $p + 0.05p = 10$ or equations in two variables such as $d = 5 + 5t$, relating two varying quantities.• In each case, apart from the variables, the numbers in the equation are given explicitly. The latter use presages the use of varibles to define functions.

In high school, things get more complicated. For example, students consider the general equation for a non-vertical line, $y = mx + b$. Here they are expected to understand that m and b are fixed for any given straight line, and that by varying m and b we obtain a whole family of straight lines. In this situation, m and b are called parameters. Of course, in an episode of mathematical work, the perspective could change; students might end up solving equa-

K.OA.1 Represent addition and subtraction with objects, fingers, mental images, drawings[2], sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

1.OA.1 Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[3]

A-CED.1 Create equations and inequalities in one variable and use them to solve problems.

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

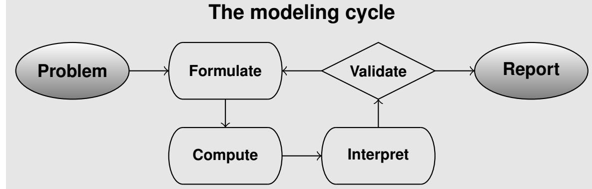
A-CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

• As noted in the Standards:

> Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems. (p. 73)

The modeling cycle



1.OA.2 Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

6.EE.6 Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

• Some writers prefer to retain the term "unknown" for the first situation and the word "variable" for the second. This is not the usage adopted in the Standards.

tions for m and b. Judging whether to explicitly indicate this—"now we will regard the parameters as variables"—or whether to ignore it and just go ahead and solve for the parameters is a matter of pedagogical judgement.

Sometimes, an equation like $y = mx + b$ is used, not to work with a parameterized family of equations, but to consider the general form of an equation and prove something about it. For example, you might want take two points (x_1, y_1) and (x_2, y_2) on the graph of $y = mx + b$ and show that the slope of the line they determine is m. In this situation you might refer to m and b as constants rather than as parameters.

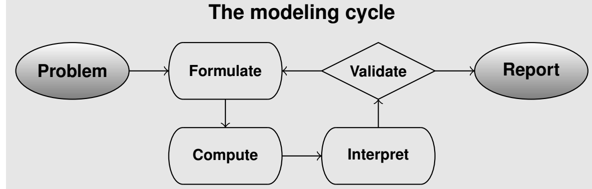
Finally, there are situations where an equation is used to describe the relationship between a number of different quantities, to which none of these terms apply.[A-CED.4] For example, Ohm's Law $V = IR$ relates the voltage, current, and resistance of an electrical circuit. An equation used in this way is sometimes called a formula. It is perhaps best to avoid using the terms "variable," "parameter," or "constant" when working with this formula, because there are six different ways it can be viewed as defining one quantity as a function of the other with a third held constant.

A-CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Different curricular implementations of the Standards might navigate these terminological shoals in different ways (that might include trying to avoid them entirely).

Modeling with equations Consider the *Formulate* node in the modeling cycle. In elementary school, students formulate equations to solve word problems. They begin with situations that can be represented by "situation equations" that are also "solution equations." These situations and their equations have two important characteristics. First, the actions in the situations can be straightforwardly represented by operations. For example, the action of putting together is readily represented by addition (e.g., "There were 2 bunnies and 3 more came, how many were there?"), but representing an additive comparison ("There were 2 bunnies, more came. Then there were 5. How many more bunnies came?") requires a more sophisticated understanding of addition. Second, the equations lead directly to a solution, e.g., they are of the form $2 + 3 = \square$ with the unknown isolated on one side of the equation rather than $2 + \square = 5$ or $5 - \square = 2$. More comprehensive understanding of the operations (e.g., understanding subtraction as finding an unknown addend) allows students to transform the latter types of situation equations into solution equations, first for addition and subtraction equations, then for multiplication and division equations.

The modeling cycle



In high school, there is again a difference between directly representing the situation and finding a solution. For example, in solving

> Selina bought a shirt on sale that was 20% less than the original price. The original price was \$5 more than

the sale price. What was the original price? Explain or show work.

students might let p be the original price in dollars and then express the sale price in terms of p in two different ways and set them equal. On the one hand, the sale price is 20% less than the original price, and so equal to $p - 0.2p$. On the other hand, it is \$5 less than the original price, and so equal to $p - 5$. Thus they want to solve the equation

$$p - 0.2p = p - 5.$$

In this task, the formulation of the equation tracks the text of the problem fairly closely, but requires more than a direct representation of "The original price was \$5 more than the sale price." To obtain an expression for the sale price, this sentence needs to be reinterpreted as "the sale price is \$5 less than the original price." Because the words "less" and "more" have often traditionally been the subject of schemes for guessing the operation required in a problem without reading it, this shift is significant, and prepares students to read more difficult and realistic task statements.

Indeed, in a high school modeling problem, there might be significantly different ways of going about a problem depending on the choices made, and students must be much more strategic in formulating the model.

For example, students enter high school understanding a solution of an equation as a number that satisfies the equation[6.EE.6] rather than as the outcome of an accepted series of manipulations for a given type of equation. Such an understanding is a first step in allowing students to represent a solution as an unknown number and to describe its properties in terms of that representation.

The *Compute* node of the modeling cycle is dealt with in the next section, on solving equations.

The *Interpret* node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied.

The *Validate* node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself ("Model with mathematics," MP.4).

[6.EE.6] Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

Formulating an equation by checking a solution

Mary drives from Boston to Washington, and she travels at an average rate of 60 mph on the way down and 50 mph on the way back. If the total trip takes $18\frac{1}{3}$ hours, how far is Boston from Washington?

Commentary How can we tell whether or not a specific number of miles s is a solution for this problem? Building on the understanding of rate, time, and distance developed in Grades 7 and 8, students can check a proposed solution s, e.g., 500 miles. They know that the time required to drive down is $\frac{500}{60}$ hours and to drive back is $\frac{500}{50}$ hours. If 500 miles is a solution, the total time $\frac{500}{60} + \frac{500}{50}$ should be $18\frac{1}{3}$ hours. This is not the case. How would we go about checking another proposed solution, say, 450 miles? Now the time required to drive down is $\frac{450}{60}$ hours and to drive back is $\frac{450}{50}$ hours.
Formulating these repeated calculations be formulated in terms of s rather than a specific number (MP.8), leads to the equation $\frac{s}{60} + \frac{s}{50} = 18\frac{1}{3}$.

Task and discussion adapted from Cuoco, 2008, "Introducing Extensible Tools in High School Algebra," in *Algebra and Algebraic Thinking in School Mathematics*, National Council of Teachers of Mathematics.

Reasoning with Equations and Inequalities

Equations in one variable A naked equation, such as $x^2 = 4$, without any surrounding text, is merely a sentence fragment, neither true nor false, since it contains a variable x about which nothing is said. A written sequence of steps to solve an equation, such as in the margin, is code for a narrative line of reasoning using words like "if," "then," "for all," and "there exists." In the process of learning to solve equations, students learn certain standard "if–then" moves, for example "if $x = y$ then $x+2 = y+2$." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in the standards in this domain is that students understand that solving equations is a process of reasoning.[A-REI.1] This does not necessarily mean that they always write out the full text; part of the advantage of algebraic notation is its compactness. Once students know what the code stands for, they can start writing in code. Thus, eventually students might go from $x^2 = 4$ to $x = \pm 2$ without intermediate steps.[4]

Understanding solving equations as a process of reasoning demystifies "extraneous" solutions that can arise under certain solution procedures.[A-REI.2] The reasoning begins from the assumption that x is a number that satisfies the equation and ends with a list of possibilities for x. But not all the steps are necessarily reversible, and so it is not necessarily true that every number in the list satisfies the equation. For example, it is true that if $x = 2$ then $x^2 = 4$. But it is not true that if $x^2 = 4$ then $x = 2$ (it might be that $x = -2$). Squaring both sides of an equation is a typical example of an irreversible step; another is multiplying both sides of the equation by a quantity that might be zero.

With an understanding of solving equations as a reasoning process, students can organize the various methods for solving different types of equations into a coherent picture. For example, solving linear equations involves only steps that are reversible (adding a constant to both sides, multiplying both sides by a non-zero constant, transforming an expression on one side into an equivalent expression). Therefore solving linear equations does not produce extraneous solutions.[A-REI.3] The process of completing the square also involves only this same list of steps, and so converts any quadratic equation into an equivalent equation of the form $(x-p)^2 = q$ that has exactly the same solutions.[A-REI.4a] The latter equation is easy to solve by the reasoning explained above.

This example sets up a theme that reoccurs throughout algebra; finding ways of transforming equations into certain standard forms that have the same solutions. For example, an exponential equation of the form $c \cdot d^{kx} =$ constant can be transformed into one of the form

[4] It should be noted, however, that calling this action "taking the square root of both sides" is dangerous, because it suggests the erroneous statement $\sqrt{4} = \pm 2$.

Fragments of reasoning

$$\begin{aligned} x^2 &= 4 \\ x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= 2, -2 \end{aligned}$$

This sequence of equations is short-hand for a line of reasoning:

> If x is a number whose square is 4, then $x^2 - 4 = 0$. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers x, it follows that $(x-2)(x+2) = 0$. So either $x - 2 = 0$, in which case $x = 2$, or $x + 2 = 0$, in which case $x = -2$.

More might be said: a justification of the last step, for example, or a check that 2 and -2 actually do satisfy the equation, which has not been proved by this line of reasoning.

A-REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

A-REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

A-REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.4a Solve quadratic equations in one variable.

a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

$b^x = a$, the solution to which is (by definition) a logarithm. Students obtain such solutions for specific cases[F-LE.4] and those intending study of advanced mathematics understand these solutions in terms of the inverse relationship between exponents and logarithms.[F-BF.5]

It is traditional for students to spend a lot of time on various techniques of solving quadratic equations, which are often presented as if they are completely unrelated (factoring, completing the square, the quadratic formula). In fact, as we have seen, the key step in completing the square involves at its heart factoring. And the quadratic formula is nothing more than an encapsulation of the method of completing the square, expressing the actions repeated in solving a collection of quadratic equations with numerical coefficients with a single formula (MP.8). Rather than long drills on techniques of dubious value, students with an understanding of the underlying reasoning behind all these methods are opportunistic in their application, choosing the method that best suits the situation at hand.[A-REI.4b]

Systems of equations Student work with solving systems of equations starts the same way as work with solving equations in one variable; with an understanding of the reasoning behind the various techniques.[A-REI.5] An important step is realizing that a solution to a system of equations must be a solution all of the equations in the system simultaneously. Then the process of adding one equation to another is understood as "if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum of the left sides of the two equations is equal to the sum of the right sides." Since this reasoning applies equally to subtraction, the process of adding one equation to another is reversible, and therefore leads to an equivalent system of equations.

Understanding these points for the particular case of two equations in two variables is preparation for more general situations. Such systems also have the advantage that a good graphical visualization is available; a pair (x, y) satisfies two equations in two variables if it is on both their graphs, and therefore an intersection point of the graphs.[A-REI.6]

Another important method of solving systems is the method of substitution. Again this can be understood in terms of simultaneity; if (x, y) satisfies two equations simultaneously, then the expression for y in terms of x obtained from the first equation should form a true statement when substituted into the second equation. Since a linear equation can always be solved for one of the variables in it, this is a good method when just one of the equations in a system is linear.[A-REI.7]

+ In more advanced courses, students see systems of linear equa-
+ tions in many variables as single matrix equations in vector vari-
+ ables.[A-REI.8, A-REI.9]

F-LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

F-BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

A-REI.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

A-REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

A-REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

A-REI.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).

Visualizing solutions graphically Just as the algebraic work with equations can be reduced to a series of algebraic moves unsupported by reasoning, so can the graphical visualization of solutions. The simple idea that an equation $f(x) = g(x)$ can be solved (approximately) by graphing $y = f(x)$ and $y = g(x)$ and finding the intersection points involves a number of pieces of conceptual understanding.[A-REI.11] This seemingly simple method, often treated as obvious, involves the rather sophisticated move of reversing the reduction of an equation in two variables to an equation in one variable. Rather, it seeks to convert an equation in one variable, $f(x) = g(x)$, to a system of equations in two variables, $y = f(x)$ and $y = g(x)$, by introducing a second variable y and setting it equal to each side of the equation. If x is a solution to the original equation then $f(x)$ and $g(x)$ are equal, and thus (x, y) is a solution to the new system. This reasoning is often tremendously compressed and presented as obvious graphically; in fact following it graphically in a specific example can be instructive.

Fundamental to all of this is a simple understanding of what a graph of an equation in two variables means.[A-REI.10]

A-REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

A-REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Progressions for the Common Core State Standards in Mathematics (draft)

21 April 2012

High School Statistics and Probability★

Overview

In high school, students build on knowledge and experience described in the 6-8 Statistics and Probability Progression. They develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. Students learn that formal inference procedures are designed for studies in which the sampling or assignment of treatments was random, and these procedures may not be informative when analyzing non-randomized studies, often called observational studies. For example, a random selection of 100 students from your school will allow you to draw some conclusion about all the students in the school, whereas taking your class as a sample will not allow that generalization.

Probability is still viewed as long-run relative frequency but the emphasis now shifts to conditional probability and independence, and basic rules for calculating probabilities of compound events. In the plus standards• are the Multiplication Rule, probability distributions and their expected values. Probability is presented as an essential tool for decision-making in a world of uncertainty.

In the high school Standards, individual modeling standards are indicated by a star symbol (★). Because of its strong connection with modeling, the domain of Statistics and Probability is starred, indicating that all of its standards are modeling standards.

• Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by (+).

Interpreting categorical and quantitative data

Summarize, represent, and interpret data on a single count or measurement variable Students build on the understanding of key ideas for describing distributions—shape, center, and spread—described in the Grades 6-8 Statistics and Probability Progression. This enhanced understanding allows them to give more precise answers to deeper questions, often involving comparisons of data sets. Students use shape and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. They also add a key measure of variation to their toolkits.

In connection with the mean as a measure of center, the *standard deviation* is introduced as a measure of variation. The standard deviation is based on the squared deviations from the mean, but involves much the same principle as the mean absolute deviation (MAD) that students learned about in Grades 6-8. Students should see that the standard deviation is the appropriate measure of spread for data distributions that are approximately normal in shape, as the standard deviation then has a clear interpretation related to relative frequency.

The margin shows two ways of comparing height data for males and females in the 20-29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called *parallel* (or *side-by-side*) *box plots* and *parallel histograms*.[S-ID.1] The parallel box plots show an obvious difference in the medians and the IQRs for the two groups; the medians for males and females are, respectively, 71 inches and 65 inches, while the IQRs are 4 inches and 5 inches. Thus, male heights center at a higher value but are slightly more variable.

The parallel histograms show the distributions of heights to be mound shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it just from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean.[S-ID.2,S-ID.3] They should also observe that the two measures of center, median and mean, tend to be close to each other for symmetric distributions.

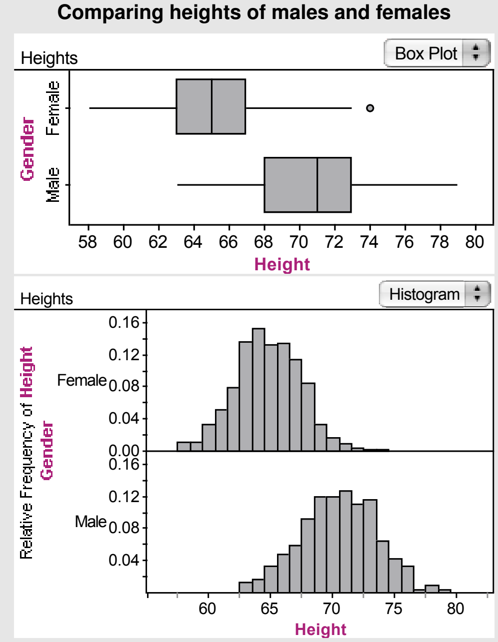
Data on heights of adults are available for anyone to look up. But how can we answer questions about standardized test scores when individual scores are not released and only a description of the distribution of scores is given? Students should now realize that we can do this only because such standardized scores generally have

Shapes of distributions

In Grade 6, students began to distinguish among distributions that were skewed or approximately symmetric. In high school, they distinguish among the latter, according to whether or not they are approximately normal, that is approximately bell-shaped.

S-ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

Comparing heights of males and females



Heights of U.S. males and females in the 20–29 age group. Source: *U.S. Census Bureau, Statistical Abstract of the United States: 2009*, Table 201.

S-ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.

S-ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

a distribution that is mound-shaped and somewhat symmetric, i.e., approximately normal.• For example, SAT math scores for a recent year have a mean of 516 and a standard deviation of 116.• Thus, about 16% of the scores are above 632. In fact, students should be aware that technology now allows easy computation of any area under a normal curve. "If Alicia scored 680 on this SAT mathematics exam, what proportion of students taking the exam scored less than she scored?" (Answer: about 92%.)[S-ID.4]

Summarize, represent, and interpret data on two categorical and quantitative variables As with univariate data analysis, students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data.[MP7, MP4]

The table below shows statistics from the Center for Disease Control relating HIV risk to age groups. Students should be able to explain the meaning of a row or column total (marginal), a row or column percentage (conditional) or a "total" percentage (joint). They should realize that possible associations between age and HIV risk are best explained in terms of the row or column conditional percentages. Are the comparisons of percentages valid when the first age category is much smaller (in years) than the others?[S-ID.5]

HIV risk by age groups, in percent of population

	Age	18–24	25–44	45–64	Row Total
Not at risk	Row %	14.0	59.6	26.4	100.0
	Column %	35.0	51.7	27.2	
	Total %	5.6	23.6	10.5	39.6
At risk	Row %	17.1	36.5	46.4	100.0
	Column %	65.0	48.3	72.8	
	Total %	10.3	22.0	28.1	60.4
Column total	Row %	15.9	45.6	38.5	100.0
	Column %	100.0	100.0	100.0	100.0
	Total %	15.9	45.6	38.5	100.0

Source: Center for Disease Control,
`http://apps.nccd.cdc.gov/s_broker/WEATSQL.exe/weat/freq_year.hsql`

Students have seen scatter plots in Grade 8 and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model tells us about that relationship. Some of the data should come from science, as in the examples about cricket chirps and temperature, and tree growth and age, and some from other aspects of their everyday life, e.g., cost of pizza and calories per slice (p. 6).

If you have a keen ear and some crickets, can the cricket chirps help you predict the temperature? The margin shows data modeled in a scientific investigation of that phenomenon. In this situation, the variables have been identified as chirps per second and temperature in degrees Fahrenheit. The cloud of points in the scatter plot is essentially linear with a moderately strong positive relationship. It looks like there must be something other than random behavior in

• At this level, students are not expected to fit normal curves to data. (In fact, it is rather complicated to rescale data plots to be density plots and then find the best fitting curve.) Instead, the aim is to look for broad approximations, with application of the rather rough "empirical rule" (also called the 68%–95% Rule) for distributions that are somewhat bell-shaped. The better the bell, the better the approximation. Using such approximations is partial justification for the introduction of the standard deviation.

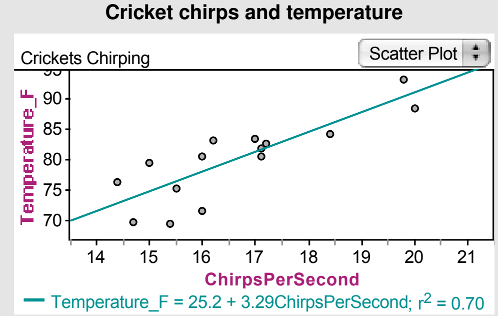
• See http://professionals.collegeboard.com/profdownload/2010-total-group-profile-report-cbs.pdf.

S-ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

MP7, MP4 Looking for patterns in tables and on scatter plots; modeling patterns in scatter plots with lines.

S-ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

Cricket chirps and temperature



Source: George W. Pierce, *The Songs of Insects*, Harvard University Press, 1949, pp. 12–21.

this association. A model has been formulated: The least squares regression line• has been fit by technology.[S-ID.6] The model is used to draw conclusions: The line estimates that, on average, each added chirp predicts an increase of about 3.29 degrees Fahrenheit.

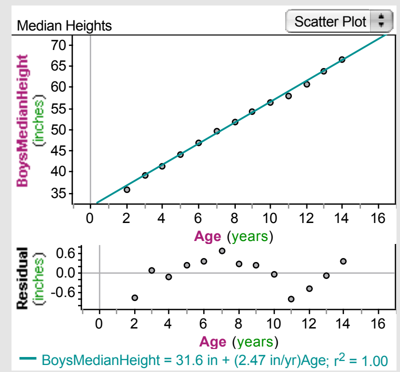
But, students must learn to take a careful look at scatter plots, as sometimes the "obvious" pattern does not tell the whole story, and can even be misleading. The margin shows the median heights of growing boys through the ages 2 to 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit. [S-ID.6c] But, the *residuals*, the collection of differences between the corresponding coordinate on the least squares line and the actual data value for each age, reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.[S-ID.6b] What would be a better description of the growth pattern?

It is readily apparent to students, after a little experience with plotting bivariate data, that not all the world is linear. The figure below shows the diameters (in inches) of growing oak trees at various ages (in years). A careful look at the scatter plot reveals some curvature in the pattern,[S-ID.6a] which is more obvious in the residual plot, because the older and larger trees add to the diameter more slowly. Perhaps a curved model, such as a quadratic, will fit the data better than a line. The figure below shows that to be the case.

Would it be wise to extrapolate the quadratic model to 50-year-old trees? Perhaps a better (and simpler) model can be found by thinking in terms of cross-sectional area, rather than diameter, as the measure that might grow linearly with age.[S-ID.6a] Area is proportional to the square of the diameter, and the plot of diameter squared versus age in the margin does show remarkable linearity,[S-ID.6a] but there is always the possibility of a closer fit, that students familiar with cube root, exponential, and logarithmic functions[F-IF.7] could investigate. Students should be encouraged to think about the relationship between statistical models and the real world, and how knowledge of

• This term is used to identify the line in this Progression. Students will identify the line as the "line of best fit" obtained by technology and should not be required to use or learn "least squares regression line."

Median heights of boys



Source: National Health and Nutrition Examination Survey, 2002, `www.cdc.gov/nchs/about/major/nhanes/datatblelink.htm`.

S-ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

a Fit a function to the data; use functions fitted to data to solve problems in the context of the data.

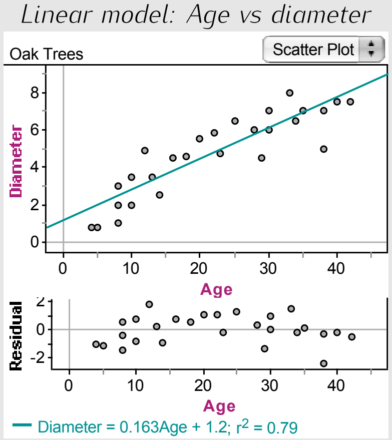
b Informally assess the fit of a function by plotting and analyzing residuals.

c Fit a linear function for a scatter plot that suggests a linear association.

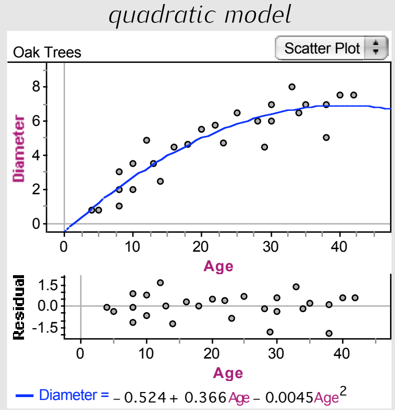
F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Three iterations of the modeling cycle

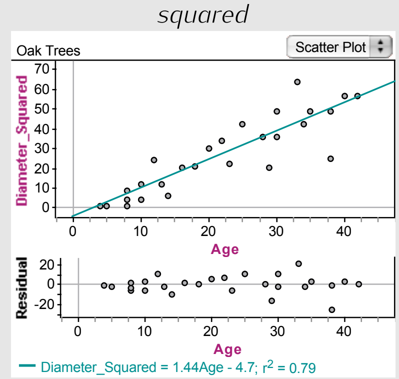
Linear model: Age vs diameter



A closer fit: Age vs diameter in a quadratic model



A simpler model: Age vs diameter squared



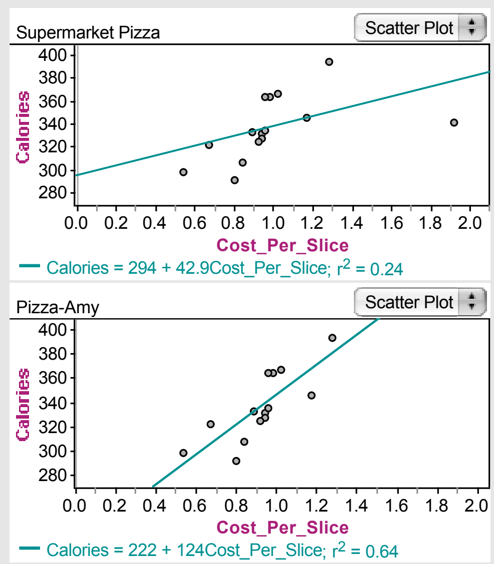
the context is essential to building good models.

Interpret linear models Students understand that the process of fitting and interpreting models for discovering possible relationships between variables requires insight, good judgment and a careful look at a variety of options consistent with the questions being asked in the investigation.[MP6]

MP6 Reasoning abstractly but quantitatively in discovering possible associations between numerical variables.

Suppose you want to see if there is a relationship between the cost per slice of supermarket pizzas and the calories per serving. The margin shows data for a sample of 15 such pizza brands, and a somewhat linear trend. A line fitted via technology might suggest that you should expect to see an increase of about 43 calories if you go from one brand to another that is one dollar more in price. But, the line does not appear to fit the data well and the correlation coefficient r (discussed below) is only about 0.5. Students will observe that there is one pizza that does not seem to fit the pattern of the others, the one with maximum cost. Why is it way out there? A check reveals that it is Amy's Organic Crust & Tomatoes, the only organic pizza in the sample. If the outlier (Amy's pizza) is removed and the discussion is narrowed to non-organic pizzas (as shown in the plot for pizzas other than Amy's), the relationship between calories and price is much stronger with an expected increase of 124 calories[S-ID.7] per extra dollar spent and a correlation coefficient of 0.8. Narrowing the question allows for a better interpretation of the slope of a line fitted to the data.[S-ID.8]

Pizza: Calories per slice vs cost



Source: *Consumer Reports*, January 2002.

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

S-ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.

The *correlation coefficient* measures the "tightness" of the data points about a line fitted to data, with a limiting value of 1 (or -1) if all points lie precisely on a line of positive (or negative) slope. For the line fitted to cricket chirps and temperature (p. 4), the correlation is 0.84, and for the line fitted to boys' height (p. 5), it is about 1.0. However, the quadratic model for tree growth (p. 5) is non-linear, so the value of its correlation coefficient has no direct interpretation.[S-ID.8] (The square of the correlation coefficient, however, does have an interpretation for such models.)

In situations where the correlation coefficient of a line fitted to data is close to 1 or -1, the two variables in the situation are said to have a *high correlation*. Students must see that one of the most common misinterpretations of correlation is to think of it as a synonym for causation. A high correlation between two variables (suggesting a statistical association between the two) does *not* imply that one causes the other. It is not a cost increase that causes calories to increase in pizza, and it is not a calorie increase per se that causes cost to increase; the addition of other expensive ingredients cause both to increase simultaneously.[S-ID.9] Students should look for examples of correlation being interpreted as cause and sort out why that reasoning is incorrect (MP3). Examples may include medications versus disease symptoms and teacher pay or class size versus high school graduation rates. One good way of establishing cause

S-ID.9 Distinguish between correlation and causation.

is through the design and analysis of randomized experiments, and that subject comes up in the next section.

Making inferences and justifying conclusions

Understand and evaluate random processes underlying statistical experiments Students now move beyond analyzing data to making sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter; choose a probability model for collecting data relevant to that parameter; collect data; compare the results seen in the data with what is expected under the hypothesis. If the observed results are far away from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability.[S-IC.1]

S-IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

But, what is considered "low"? That determination is left to the investigator and the circumstances surrounding the decision to be made. Statistics and probability weigh the chances; the person in charge of the investigation makes the final choice. (This is much like other areas of life in which the teacher or physician weighs the evidence and provides your chances of passing a test or easing certain disease symptoms; you make the choice.)

Consider this example. You cannot seem to roll an even number with a certain number cube. The statistical question is, "Does this number cube favor odd numbers?" The hypothesis is, "This cube does not favor odd numbers," which is the same as saying that the proportion of odd numbers rolled, in the long run, is 0.5, or the probability of tossing an odd number with this cube is 0.5. Then, toss the cube and collect data on the observed number of odds. Suppose you get an odd number in each of the:

first two tosses, which has probability $\frac{1}{4} = 0.25$ under the hypothesis;

first three tosses, which has probability $\frac{1}{8} = 0.125$ under the hypothesis;

first four tosses, which has probability $\frac{1}{16} = 0.0625$ under the hypothesis;

first five tosses, which has probability $\frac{1}{32} = 0.03125$ under the hypothesis.

At what point will students begin to seriously doubt the hypothesis that the cube does not favor odd numbers? Students should experience a number of simple situations like this to gain an understanding of how decisions based on sample data are related to probability, and that this decision process does not guarantee a correct answer to the underlying statistical question.[S-IC.3]

S-IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

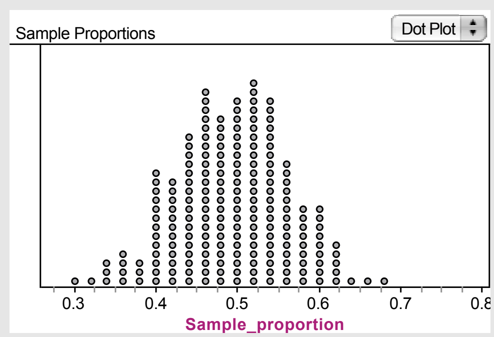
Make inferences and justify conclusions from sample surveys, experiments, and observational studies Once they see how probability intertwines with data collection and analysis, students use this knowledge to make statistical inferences from data collected in sample surveys and in designed experiments, aided by simulation and the technology that affords it.[MP5, MP3]

MP5, MP3 Using a variety of statistical tools to construct and defend logical arguments based on data.

A *Time* magazine poll reported on the status of American women. One of the statements in the poll was "It is better for a family if the father works outside the home and the mother takes care of children." Fifty-one percent of the sampled women agreed with the statement while 57% of the sampled men agreed. A note on the polling methodology states that about 1600 men and 1800 women were randomly sampled in the poll and the margin of error was about two percentage points. What is the margin of error and how is it interpreted in this context? We'll come back to the *Time* poll after exploring this question further.

"Will 50% of the homeowners in your neighborhood agree to support a proposed new tax for schools?" A student attempts to answer this question by taking a random sample of 50 homeowners in her neighborhood and asking them if they support the tax. Twenty of the sampled homeowners say they will support the proposed tax, yielding a sample proportion of $\frac{20}{50} = 0.4$. That seems like bad news for the schools, but could the population proportion favoring the tax in this neighborhood still be 50%? The student knows that a second sample of 50 homeowners might produce a different sample proportion and wonders how much variation there might be among sample proportions for samples of size 50 if, in fact, 50% is the true population proportion. Having a graphing calculator available, she simulates this sampling situation by repeatedly drawing random samples of size 50 from a population of 50% ones and 50% zeros, calculating and plotting the proportion of ones observed in each sample. The result for 200 trials is displayed in the margin. The simulated values at or below the observed 0.4 number 25 out of 200, or $\frac{25}{200} = 0.125$. So, the chance of seeing a 40% or fewer favorable response in the sample even if the true proportion of such responses was 50% is not all that small, casting little doubt on 50% as a plausible population value.

Proportions in random samples of size 50



Relating the components of this example to the statistical reasoning process, students see that the hypothesis is that the population parameter is 50% and the data are collected by a random sample. The observed sample proportion of 40% was found to be not so far from the 50% so as to cause serious doubt about the hypothesis. This lack of doubt was justified by simulating the sampling process many times and approximating the chance of a sample proportion being 40% or less under the hypothesis.[MP8]

MP8 Observing regular patterns in distributions of sample statistics.

Students now realize that there are other plausible values for the population proportion, besides 50%. The plot of the distribution of sample proportions in the margin is mound-shaped (approximately normal) and somewhat symmetric with a mean of about 0.49 (close to 0.50) and a standard deviation of about 0.07. From knowledge of the normal distribution,[S-ID.4] students knows that about 95% of the possible sample proportions that could be generated this way

S-ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

will fall within two standard deviations of the mean. This two-standard deviation distance is called the *margin of error* for the sample proportions. In this example with samples of size 50, the margin of error is $2 \cdot 0.07 = 0.14$.

Suppose the true population proportion is 0.60. The distribution of the sample proportions will still look much like the plot in the margin, but the center of the distribution will be at 0.60. In this case, the observed sample proportion 0.4 will not be within the margin of error. Reasoning this way leads the student to realize that any population proportion in the interval 0.40 ± 0.14 will result in the observed sample proportion of 0.40 being within the middle 95% of the distribution of sample proportions, for samples of size 50. Thus, the interval

$$\text{observed sample proportion} \pm \text{margin of error}$$

includes the plausible values for the true population proportion in the sense that any of those populations would have produced the observed sample proportion within its middle 95% of possible outcomes. In other words, the student is confident that the proportion of homeowners in her neighborhood that will favor the tax is between 0.26 and 0.54.[S-IC.4] All of this depends on random sampling because the characteristics of distributions of sample statistics are predictable only if the sampling is random.

S-IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

With regard to the *Time* poll on the status of women, the student now sees that the plausible proportions of men who agree with the statement lie between 55% and 59% while the plausible proportions of women who agree lie between 49% and 53%. What interesting conclusions might be drawn from this?[S-IC.6]

S-IC.6 Evaluate reports based on data.

Students' understanding of random sampling as the key that allows the computation of margins of error in estimating a population quantity can now be extended to the random assignment of treatments to available units in an experiment. A clinical trial in medical research, for example, may have only 50 patients available for comparing two treatments for a disease. These 50 are the population, so to speak, and randomly assigning the treatments to the patients is the "fair" way to judge possible treatment differences, just as random sampling is a fair way to select a sample for estimating a population proportion.

There is little doubt that caffeine stimulates bodily activity, but how much does it take to produce a significant effect? This is a question that involves measuring the effect of two or more treatments and deciding if the different interventions have differing effects. To obtain a partial answer to the question on caffeine, it was decided to compare a treatment consisting of 200 mg of caffeine with a control of no caffeine in an experiment involving a finger tapping exercise.

Twenty male students were randomly assigned to one of two treatment groups of 10 students each, one group receiving 200 milligrams of caffeine and the other group no caffeine. Two hours later

Finger taps per minute in a caffeine experiment

	0 mg caffeine	200 mg caffeine
	242	246
	245	248
	244	250
	248	252
	247	248
	248	250
	242	246
	244	248
	246	245
	242	250
Mean	244.8	248.3

Source: Draper and Smith, *Applied Regression Analysis*, John Wiley and Sons, 1981 the students were given a finger tapping exercise. The response is the number of taps per minute, as shown in the table.

The plot of the finger tapping data shows that the two data sets tend to be somewhat symmetric and have no extreme data points (outliers) that would have undue influence on the analysis. The sample mean for each data set, then, is a suitable measure of center, and will be used as the statistic for comparing treatments.

The mean for the 200 mg data is 3.5 taps larger than that for the 0 mg data. In light of the variation in the data, is that enough to be confident that the 200 mg treatment truly results in more tapping activity than the 0 mg treatment? In other words, could this difference of 3.5 taps be explained simply by the randomization (the luck of the draw, so to speak) rather than any real difference in the treatments? An empirical answer to this question can be found by "re-randomizing" the two groups many times and studying the distribution of differences in sample means. If the observed difference of 3.5 occurs quite frequently, then we can safely say the difference could simply be due to the randomization process. If it does not occur frequently, then we have evidence to support the conclusion that the 200 mg treatment has increased mean finger tapping count.

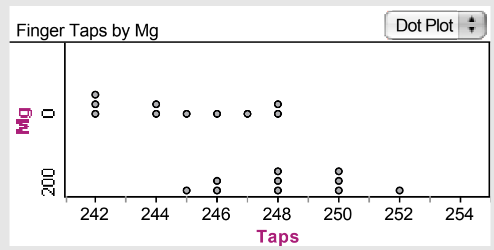
The re-randomizing can be accomplished by combining the data in the two columns, randomly splitting them into two different groups of ten, each representing 0 and 200 mg, and then calculating the difference between the sample means. This can be expedited with the use of technology.

The margin shows the differences produced in 400 re-randomizations of the data for 200 and 0 mg. The observed difference of 3.5 taps is equaled or exceeded only once out of 400 times. Because the observed difference is reproduced only 1 time in 400 trials, the data provide strong evidence that the control and the 200 mg treatment do, indeed, differ with respect to their mean finger tapping counts. In fact, we can conclude with little doubt that the caffeine is the *cause* of the increase in tapping because other possible factors should have been balanced out by the randomization.[S-IC.5] Students should be able to explain the reasoning in this decision and the nature of the error that may have been made.

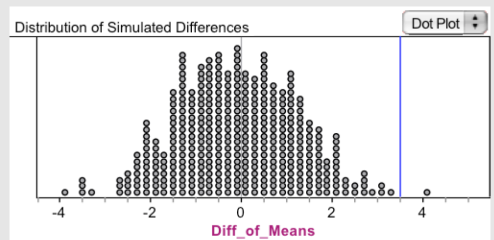
It must be emphasized repeatedly that the probabilistic reasoning underlying statistical inference is introduced into the study by way of random sampling in sample surveys and random assignment of treatments in experiments. No randomization, no such reasoning! Students will know, however, that randomization is not possible in many types of statistical investigations. Society will not condone the assigning of known harmful "treatments" (smoking, for example) to patients, so studies of the effects of smoking on health cannot be randomized experiments. Such studies must come from *observing* people who choose to smoke, as compared to those who do not, and are, therefore, called *observational studies*. The oak tree study (p. 5) and the pizza study (p. 6) are both observational studies.

Surveys of samples to estimate population parameters, random-

Plot of finger tapping data



Differences in re-randomized means for finger tapping data



S-IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

ized experiments to compare treatments and show cause, and observational studies to indicate possible associations among variables are the three main methods of data production in statistical studies. Students should understand the distinctions among these three and practice perceiving them in studies that are reported in the media, deciding if appropriate inferences seem to have been drawn.S-IC.3

S-IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

Conditional probability and the rules of probability

In Grades 7 and 8, students encountered the development of basic probability, including chance processes, probability models, and sample spaces. In high school, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value. As seen in the making inferences section above, there is a strong connection between statistics and probability. This will be seen again in this section with the use of data in selecting values for probability models.

Understand independence and conditional probability and use them to interpret data In developing their understanding of conditional probability and independence, students should see two types of problems, one in which the uniform probabilities attached to outcomes leads to independence and one in which it does not. For example, suppose a student is randomly guessing the answers to all four true–false questions on a quiz. The outcomes in the sample space can be arranged as shown in the margin.[S-CP.1] Probabilities assigned to these outcomes should be equal because random guessing implies that no one outcome should be any more likely than another.

By simply counting equally likely outcomes,

$$\text{P(exactly}^{\text{MP6}}\text{ two correct answers)} = \frac{6}{16}$$

and

$$\begin{aligned}\text{P(at least one correct answer)} &= \frac{15}{16}\\ &= 1 - \text{P(no correct answers)}.\end{aligned}$$

Likewise,

$$\begin{aligned}\text{P(C on first question)} &= \frac{1}{2}\\ &= \text{P(C on second question)}\end{aligned}$$

as should seem intuitively reasonable. Now,

$$\begin{aligned}P[\text{(C on first question) and (C on second question)}] &= \frac{4}{16}\\ &= \frac{1}{4}\\ &= \frac{1}{2}\cdot\frac{1}{2},\end{aligned}$$

Possible outcomes: Guessing on four true–false questions

Number correct	Outcomes	Number correct	Outcomes	Number correct	Outcomes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

S-CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

MP6 Attend to precision. "Two correct answers" may be interpreted as "at least two" or as "exactly two."

which shows that the two events (C on first question) and (C on second question) are independent, by the definition of independence.• This, too, should seem intuitively reasonable to students because the random guess on the second question should not have been influenced by the random guess on the first.

• Two events A and B are said to be independent if $P(A) \cdot P(B) = P(A \text{ and } B)$.

Students may contrast the quiz scenario above with the scenario of choosing at random two students to be leaders of a five-person working group consisting of three girls (*A*pril, *B*riana, and *C*yndi) and two boys (*D*aniel and *E*rnesto). The first name chosen indicates the discussion leader and the second the recorder, so order of selection is important. The 20 outcomes are displayed in the margin.

Selecting two students from three girls and two boys

Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Here, the probability of selecting two girls is:

$$\begin{aligned} \text{P(two girls selected)} &= \frac{6}{20} \\ &= \frac{3}{10} \end{aligned}$$

whereas

$$\begin{aligned} \text{P(girl selected on first draw)} &= \frac{12}{20} \\ &= \frac{3}{5} \\ &= \text{P(girl selected on second draw).} \end{aligned}$$

Because $\frac{3}{5} \cdot \frac{3}{5} \neq \frac{3}{10}$, these two events are not independent. The selection of the second person does depend on the selection of the first when the same person cannot be selected twice.

Another way of viewing independence is to consider the conditional probability of an event A given an event B, P(A|B), as the probability of A in the sample space restricted to just those outcomes that constitute B. In the table of outcomes for guessing on the true-false questions,

$$\begin{aligned} \text{P(C on second question} \mid \text{C on first question)} &= \frac{4}{8} \\ &= \frac{1}{2} \\ &= \text{P(C on second)} \end{aligned}$$

and students see that knowledge of what happened on the first question does not alter the probability of the outcome on the second; the two events are independent.

In the selecting students scenario, the conditional probability of a girl on the second selection, given that a girl was selected on the first is

$$\begin{aligned} \text{P(girl on second} \mid \text{girl on first)} &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\text{P(girl on second)} = \frac{3}{5}.$$

So, these two events are again seen to be dependent. The outcome of the second draw does depend on what happened at the first draw.[S-CP.3]

Students understand that in real world applications the probabilities of events are often approximated by data about those events. For example, the percentages in the table for HIV risk by age group (p. 4) can be used to approximate probabilities of HIV risk with respect to age or age with respect to HIV risk for a randomly selected adult from the U.S. population of adults. Emphasizing the conditional nature of the row and column percentages:

$$\text{P(adult is age 18 to 24} \mid \text{adult is at risk)} = 0.171$$

whereas

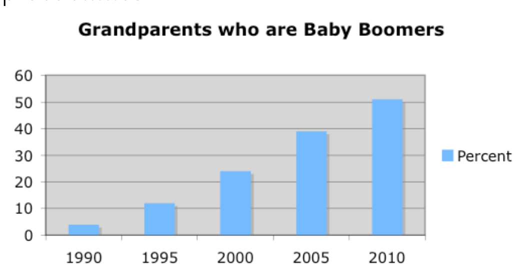
$$\text{P(adult is at risk} \mid \text{adult is age 18 to 24)} = 0.650.$$

Comparing the latter to

$$\text{P(adult is at risk} \mid \text{adult is age 25 to 44)} = 0.483$$

shows that the conditional distributions change from column to column, reflecting dependence and an association between age category and HIV risk.[S-CP.4, S-CP.5]

Students can gain practice in interpreting percentages and using them as approximate probabilities from study data presented in the popular press. Quite often the presentations are a little confusing and can be interpreted in more than one way. For example, two data summaries from *USA Today* are shown below. What might these percentages represent and how might they be used as approximate probabilities?[S-CP.5]



Top age groups for DUI	
21–25	29%
26–29	24%
18–20	20%
30–34	19%

Use the rules of probability to compute probabilities of compound events in a uniform probability model The two-way table for HIV risk by age group (p. 4) gives percentages from a data analysis that can be used to approximate probabilities, but students realize that such tables can be developed from theoretical probability models. Suppose, for example, two fair six-sided number cubes are rolled, giving rise to 36 equally likely outcomes.

S-CP.3 Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

S-CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

Outcomes for specified events can be diagramed as sections of the table, and probabilities calculated by simply counting outcomes. This type of example is one way to review information on conditional probability and introduce the addition and multiplication rules. For example, defining events:

A is "you roll numbers summing to 8 or more"

B is "you roll doubles"

and counting outcomes leads to

$$\begin{aligned} P(A) &= \frac{15}{36} \\ P(B) &= \frac{6}{36} \\ P(A \text{ and } B) &= \frac{3}{36}, \quad \text{and} \\ P(B|A) &= \frac{3}{15}, \quad \text{the fraction of A's 15 outcomes that also fall in B.}^{\text{S-CP.6}} \end{aligned}$$

Possible outcomes: Rolling two number cubes

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Now, by counting outcomes

$$P(A \text{ or } B) = \frac{18}{36}$$

or by using the Addition Rule[S-CP.7]

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= \frac{15}{36} + \frac{6}{36} - \frac{3}{36} \\ &= \frac{18}{36}. \end{aligned}$$

\+ By the Multiplication Rule[S-CP.8]

$$\begin{aligned} P(A \text{ and } B) &= P(A)P(B|A) \\ &= \frac{15}{36} \cdot \frac{3}{15} \\ &= \frac{3}{36}. \end{aligned}$$

The assumption that all outcomes of rolling each cube once are equally likely results in the outcome of rolling one cube being independent of the outcome of rolling the other.[S-CP.5] Students should understand that independence is often used as a simplifying assumption in constructing theoretical probability models that approximate real situations. Suppose a school laboratory has two smoke alarms as a built in redundancy for safety. One has probability 0.4 of going off when steam (not smoke) is produced by running hot water and the other has probability 0.3 for the same event. The probability

S-CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.

S-CP.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.

S-CP.8 (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interpret the answer in terms of the model.

S-CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

that they both go off the next time someone runs hot water in the sink can be reasonably approximated as the product $0.4 \cdot 0.3 = 0.12$, even though there may be some dependence between two systems operating in the same room. Modeling independence is much easier than modeling dependence, but models that assume independence are still quite useful.

Using probability to make decisions

Calculate expected values and use them to solve problems As students gain experience with probability problems that deal with listing and counting outcomes, they will come to realize that, most often, applied problems concern some numerical quantity of interest rather than a description of the outcomes themselves.MP1 MP2 Advertisers want to know how many customers will purchased their product, not the order in which they came into the store. A political pollster wants to know how many people are likely to vote for a particular candidate and a student wants to know how many questions he is likely to get right by guessing on a true-false quiz.

MP1 Make sense of a problem, analyzing givens, constraints, relationships, and goals.

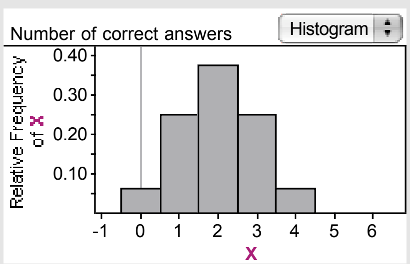
MP2 Formulate a probability model for a practical problem that reflects constraints and relationships, and reason abstractly to solve the problem.

In such situations, the outcomes can be seen as numerical values of a *random variable*.• Reconfiguring the tables of outcomes for the true-false test (p. 13) and student selection (p. 14) in a way that emphasizes these numerical values and their probabilities gives rise to the probability distributions shown below.

• Students should realize that random variables are different from the variables used in other high school domains; random variables are functions of the outcomes of a random process and thus have probabilities attached to their possible values.

True–false test

Number correct, X	Probab-ility
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$



Selection of students

Number of girls, Y	Probab-ility
0	$\frac{1}{10}$
1	$\frac{6}{10}$
2	$\frac{3}{10}$

Because probability is viewed as a long-run relative frequency, probability distributions can be treated as theoretical data distributions. If 1600 students all guessed at all four questions on the true-false test, about 400 of them would get three answers correct, about 100 four answers correct, and so on. These scores could then be averaged to come up with a mean score of:

$$0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = 2.$$

With the *number correct* labeled as X, this value is called the *expected value* of X, usually expressed as E(X). Anyone guessing at all four true-false questions on a test can expect, over the long run, to get two correct answers per test, which is intuitively reasonable.

Students then develop the general rule that, for any discrete• random variable X,

• Students need not learn the term "discrete random variable." All of the random variables treated in this Progression are discrete random variables, that is, they concern only sample spaces which are collections of discrete objects.

$$E(X) = \sum \text{(value of X)(probability of that value)}$$

where the sum extends over all values of X.S-MD.2

S-MD.2(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

For the random variable *number of girls*, Y, E(Y) = 1.2. Of course, 1.2 girls cannot be selected in any one group, but if the group selects leaders at random each day for ten days, they would be expected to choose about 12 girls as compared to 8 boys over the period.

The probability distributions considered above arise from theoretical probability models, but they can also come from empirical approximations. The margin displays the distribution of family sizes in the U.S., according to the Census Bureau. (Very few families have more than seven members.) These proportions calculated from census counts can serve as to approximate probabilities that families of given sizes will be selected in a random sample. If an advertiser randomly samples 1000 families for a special trial of a new product to be used by all members of the family, she would expect to have the product used by about 3.49 people per family, or about 3,490 people over all.

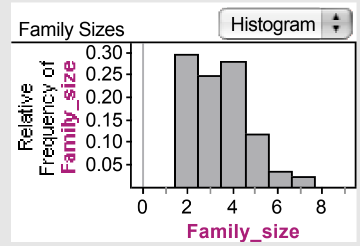
Use probability to evaluate outcomes of decisions Students should understand that probabilities and expected values must be thought of as long-term relative frequencies and means, and consider the implications of that view in decision making. Consider the following real-life example. The Wisconsin lottery had a game called "Hot Potato" that cost a dollar to play and had payoff probabilities as shown in the margin. The sum of these probabilities is not 1, but there is a key payoff value missing from the table. Students can include that key value and its probability to make this a true probability distribution and find that the expected payoff per game is about \$0.55.[S-MD.5] Losing a dollar to play the game may not mean much to an individual player, but expecting to take in \$450 for every \$1000 spent on the game means a great deal to the Wisconsin Lottery Commission!

Studying the behavior of games of chance is fun, but students must see more serious examples such as this one, based on empirical data. In screening for HIV by use of both the ELISA and Western Blot tests, HIV-positive males will test positive in 99.9% of the cases and HIV-negative males will test negative in 99.99% of the cases. Among men with low-risk behavior, the rate of HIV is about 1 in 10,000. What is the probability that a low-risk male who tests positive actually is HIV positive?

Having students turn the given rates into expected counts and placing the counts in an appropriate table is a good way for them to construct a meaningful picture of what is going on here. There are two variables, whether or not a tested person is HIV positive and whether or not the test is positive. Starting with a cohort of 10,000 low-risk males, the table might look like the one in the margin. The conditional probability of a randomly selected male being HIV positive, given that he tested positive is about 0.5! Students should discuss the implications of this in relation to decisions concerning mass screening for HIV.[S-MD.6, S-MD.7]

Empirical distribution of family size

Family size	Proportion
2	0.28
3	0.25
4	0.27
5	0.13
6	0.04
7	0.03



Source: U.S. Census Bureau, `http://www.census.gov/population/www/socdemo/hh-fam/cps2010.html`; Table F1

"Hot Potato" payoffs and probabilities

Payoff (\$)	Probability
1	$\frac{1}{9}$
2	$\frac{1}{13}$
3	$\frac{1}{43}$
6	$\frac{1}{94}$
9	$\frac{1}{150}$
18	$\frac{1}{300}$
50	$\frac{1}{2050}$
100	$\frac{1}{144000}$
300	$\frac{1}{180000}$
900	$\frac{1}{270000}$

For details about Hot Potato and other lotteries, see `www.wilottery.com/scratchgames/historical.aspx`.

S-MD.5 (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

HIV testing expected frequencies

	HIV+ male	HIV- male	Totals
HIV+ test result	0.999	1	1.999
HIV- test result	0.001	9,998	9,998.001
Totals	1	9,999	10,000

S-MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

S-MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Where the Statistics and Probability Progression might lead

Careers A few examples of careers that draw on the knowledge discussed in this Progression are actuary, manufacturing technician, industrial engineer or statistician, industrial engineer and production manager. The level of education required for these careers and sources of further information and examples of workplace tasks are summarized in the table below. Information about careers for statisticians in health and medicine, business and industry, and government appears on the web site of the American Statistical Association (`www.amstat.org/careers/index.cfm`).

	Education	Location of information, workplace task
Actuary	bachelors	*Ready or Not*, p. 79; `http://beanactuary.org/how/highschool/`
Manufacturing technician	associate	*Ready or Not*, p. 81
Industrial engineer or statistician	bachelors	`http://www.achieve.org/node/205`
Industrial engineer; production manager	bachelors	`http://www.achieve.org/node/620`

Source: *Ready or Not: Creating a High School Diploma That Counts*, 2004, `www.achieve.org/ReadyorNot`

College Most college majors in the sciences (including health sciences), social sciences, biological sciences (including agriculture), business, and engineering require some knowledge of statistics. Typically, this exposure begins with a non-calculus-based introductory course that would expand the empirical view of statistical inference found in this high school progression to a more general view based on mathematical formulations of inference procedures. (The Advanced Placement Statistics course is at this level.) After that general introduction, those in more applied areas would take courses in statistical modeling (regression analysis) and the design and analysis of experiments and/or sample surveys. Those heading to degrees in mathematics, statistics, economics, and more mathematical areas of engineering would study the mathematical theory of statistics and probability at a deeper level, perhaps along with more specialized courses in, say, time series analysis or categorical data analysis. Whatever their future holds, most students will encounter data in their chosen field—and lots of it. So, gaining some knowledge of both applied and theoretical statistics, along with basic skills in computing, will be a most valuable asset indeed!

Progressions for the Common Core State Standards in Mathematics (draft)

4 July 2013

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For updates and more information about the Progressions, see **http://ime.math.arizona.edu/progressions**.

For discussion of the Progressions and related topics, see the Tools for the Common Core blog: **http://commoncoretools.me**.

Modeling, High School

Introduction

Mathematical models describe situations in the world, to the surprise of many. Albert Einstein wondered, "How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?"• This points to the basic reason to model with mathematics and statistics: to understand reality. Reality might be described by a law of nature such as that governing the motion of an object dropped from a height above the ground[A-CED.2] or in terms of the height above the ground of a person riding a Ferris wheel,[F-TF.5] the unemployment rate,[N-Q.1] how people's heights vary,[S-ID.1] a risk factor for a disease,[S-ID.5] the effectiveness of a medical treatment,[S-ID.5] or the amount of money in a savings account to which periodic additions are made.[A-SSE.4] On a more sophisticated level, modeling the spread of an epidemic, assessing the security of a computer password, understanding cyclic populations of predator and prey in an ecosystem, finding an orbit for a communications satellite that keeps it always over the same spot, estimating how large an area of solar panels would be enough to power a city of a given size, understanding how global positioning systems (GPSs) work, estimating how long it would take to get to the nearest star—all can be done using mathematical modeling. A survey of how mathematics has impacted recent breakthroughs can be found in Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century.•

Mathematical modeling is fundamental to how mathematics is used in medicine, engineering, ecology, weather forecasting, oil exploration, finance and economics, business and marketing, climate modeling, designing search engines, understanding social networks, public key cryptography and cybersecurity, the space program, astronomy and cosmology, biology and genetics, criminology, using genetics to reconstruct how early humans spread over the planet, in testing and designing new drugs, in compressing images (JPEG) and music (MP3), in creating the algorithms that cell phones use to communicate, to optimize air traffic control and schedule flights, to design cars and wind turbines, to recommend which books (Amazon), music (Pandora) and movies (Netflix) an individual might like based on other things they rated highly. The range of careers for

• January 27, 1921, address to the Prussian Academy of Science, Berlin.

• 1960, "The Unreasonable Effectiveness of Mathematics," Communications in Pure and Applied Mathematics.

A-CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

S-ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

S-ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

• This report was published in 2012 by the National Academies Press and can be read online at `http://www.nap.edu/catalog.php?record_id=13373`.

which mathematical and statistical modeling are good preparation has expanded substantially in recent years, and the list continues to grow.

Mathematical and statistical models of real world situations range in complexity from objects or drawings that represent addition and subtraction situations[K.OA.2] to systems of equations that describe behaviors of natural phenomena such as fluid flow or the paths of ballistic missiles. Sometimes models give rather complete information about the situation. For example, writing total cost as the product of the unit price and the number bought is often a complete and accurate model of monetary costs. Some models do not give exact and complete information but approximations that may result from the features of the situation that are reasonably available or of most interest.[MP4] In the business world, the per item price when purchasing a large number of the same item is lower than for the price for a single item. This is important in modeling some situations but may be neglected in others. As another example, consider the linear function describing the cost of purchasing an automobile and gasoline for a number of years

$$C(t) = p + at,$$

where p is the purchase price, t is the number of years, and a is a constant based on assumptions of the cost of gasoline (per gallon), the number of miles driven per year and the fuel efficiency in miles per gallon.[F-BF.1] All of the quantities going into the constant a are estimates and likely will not be constant over time, but a more complex model of gasoline costs and expected driving habits requires information not available and perhaps unnecessary for decision-making. Further, there are costs not included—insurance and maintenance, for example—and the effect of a gasoline-powered automobile on the environment is not considered. However, the simple model may suffice to decide, say, between the purchase of a hybrid version and a gasoline version of an automobile, where the basic differences are in purchase price (hybrids may cost more) and fuel efficiency. Following the advice attributed to Einstein that, "Everything should be made as simple as possible, but not simpler,"[1] we can get good evidence to support the choice of the simple model $C(t) = p + at$, so in this situation this is "as simple as possible," but dropping the purchase price p, or the term at would delete critical information for our decision, based on cost differences. That would delete necessary information, moving us to Einstein's "simpler [than possible]."

Models mimic features of reality. These features are often selected for particular uses. For example, a road map is a model. So

[1]Probably a paraphrase of "It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience" from "On the Method of Theoretical Physics."

What is a model?

The word "model" can be used as a noun, verb, or adjective. As an adjective, "model" often signifies an ideal, as in "model student." In this progression, "model" will be a noun or a verb.

In elementary mathematics, a model might be a representation such as a math drawing or a situation equation (operations and algebraic thinking), line plot, picture graph, or bar graph (measurement), or building made of blocks (geometry). In Grades 6–7, a model could be a table or plotted line (ratio and proportional reasoning) or box plot, scatter plot, or histogram (statistics and probability). In Grade 8, students begin to use functions to model relationships between quantities.

Models are also used to understand mathematical or statistical concepts. In elementary grades, students use rows of dots or tape diagrams to represent addition and subtraction. Later they use tape diagrams, arrays, and area models to represent multiplication and division. In Grade 6 geometry, nets can represent a three-dimensional mathematical object (e.g., a prism) as well as a design for a real world object (e.g., a gingerbread house). In Grade 8, students use physical models, transparencies, or geometry software to understand congruence and similarity. In Grade 6–8 statistics, simulations help students to understand what can happen during statistical sampling.

In high school, modeling becomes more complex, building on what students have learned in K–8. Representations such as tables or scatter plots are often intermediate steps rather than the models themselves.

K.OA.2 Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

MP4 Mathematically proficient students . . . are comfortable making assumptions and approximations . . . realizing that these may need revision later.

F-BF.1 Write a function that describes a relationship between two quantities.

is a geological map. Features that are important on road maps, e.g., major highways, may not be important on a geological map. The features of the real world situation mimicked by a mathematical model fall into three categories:[2]

- Things whose effects are neglected.
- Things that affect the model but whose behavior the model is not designed to study—inputs or independent variables.
- Things that the model is designed to study—outputs or dependent variables.[3]

These features of a mathematical model are helpful to keep in mind. For example, in the cost function C above, the effect on environment, insurance costs, and maintenance costs are neglected. Inputs are cost of gasoline, miles driven per year, and fuel efficiency rate. The output, or dependent variable, is the cost .

Modeling in K–12

Modeling is critically important, but is not easy. Some idealized, simple modeling problems are needed for learning throughout K–12, but real problems easily available and solvable (perhaps with the assistance of technology). Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena as well as physical phenomena. Situations which are not modeled by simple equations can often be understood by simulation on a calculator, desktop, or laptop, a process which many students will find especially engaging because of its exploratory and open-ended nature. These tools allow for modeling complex real world situations, and most real world situations are complex.

While there is certainly no limit to the sophistication of a model or of the mathematics used in a model, the essence of modeling is often to use humble mathematics in rather sophisticated ways. For example, percentages are often crucial in modeling situations. "Distance equals rate times time" is a powerful idea that is introduced in grade 6 [cite] that nevertheless forms the basis for many useful models throughout high school and beyond. Or as another example, when high school students make an order of magnitude estimate, they may learn a great deal by using only simple multiplication and division. Likewise, statistical modeling in high school might often involve only measures of center and variability, rather than relying on a host of sophisticated statistical techniques. "Back of the envelope" modeling is one of the discipline's most powerful forms.

[2]Bender, 1978, *An Introduction to Mathematical Modeling*, John Wiley and Sons.

[3]Statistical modeling also involves relationships among variables, but the relationship may be construed as association (e.g., correlation) rather than dependency.

Many situations in the real world involve rate of change, with models that involve a differential equation. Although differential equations are not in the Standards, the interpretation of rates of change[S-ID.7] and the study of functions with base rules of growth[F-LE.1] prepares the way for the study of more sophisticated models in college. Likewise, using probability in modeling greatly extends the scope of real world situations which can be modeled.

News media accounts of topics of current interest often illustrate why modeling and understanding the models of others is important, mostly for informed citizenship. For example, probabilities often are stated in terms of odds in media accounts. Thus, to connect such accounts to school mathematics, students need to know the relationship between the two. Learning to model and understand models is enhanced by seeing the same mathematics or statistics model situations in different contexts.• Media accounts provide those varied contexts in circumstances that require critical thinking. Analyzing these accounts provides opportunities for students to maintain and deepen their understanding of modeling in high school and after graduation.[4]

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

F-LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

• For example, right triangles are a frequent model for situations that students may initially see as different mathematically, e.g., finding the length of the shadow cast by an upright pole and finding the height of a tree or building. A line fitted to a scatter plot is often used in statistics to model relationships between two measurement quantities. Risk factors are often derived from relative frequency within a single sample.

The Modeling Process

In the Standards, modeling means using mathematics or statistics to describe (i.e., model) a real world situation and deduce additional information about the situation by mathematical or statistical computation and analysis. For example, if the annual rate of inflation is assumed to be 3% and your current salary is $38,000 per year, what is an equivalent salary t years in the future? What salary is equivalent in 10 years? The model is a familiar one to many:

$$S(t) = 38{,}000(1.03^t).$$

This aspect of modeling produces information about the real world situation via the mathematical model, i.e. the real world is understood through the mathematics.

Complex models are often built hierarchically, out of simpler components which can then be artfully joined together to capture the behavior of the complex system. Certain simplifications have become standard based on historical use. For example, the consumer price index (CPI) and the cost of living index (COLI) are commonly cited measures that serve as agreed-upon proxies for important economic circumstances, substituting a single quantity for a more complicated collection of quantities that tend to move as a group. There is even an index of indexes, the index of leading economic indicators. The monthly payments required to amortize a home mortgage over 30 years are computed by summing a geometric series and manipulating the results.[A-SSE.4] Numerous political and economic debates

A-SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

[4]For further examples, see Dingman & Madison, 2010, "Quantitative Reasoning in the Contemporary World" [two-part article], *Numeracy*, `http://scholarcommons.usf.edu/numeracy/vol3/iss2/`.

center on how one measures amounts of money, that is, what units are used. Measuring amounts of money in nominal dollars (dollars-of-the-day) over periods of several years is very different from measuring in constant dollars (the dollar of a particular year). Measuring in percent of gross domestic product (GDP) is also different. Understanding what these are and how to move from one unit to the others is critical in understanding many issues important to personal prosperity and responsible citizenship.

Probability and statistical models abound in news media reports. Complex and heretofore unusual graphics are made possible by technology and in recent years the diversity of graphical models in media accounts has increased enormously. Many of these models and the situations they describe are very important for making decisions about health issues or political circumstances. Political polls model elections themselves,[S-IC.1] and skeptics decry their predictions because they are based on a small sample of all eligible voters. Lack of understanding leads to suspicion and distrust of democratic processes.

S-IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

Modeling in High School

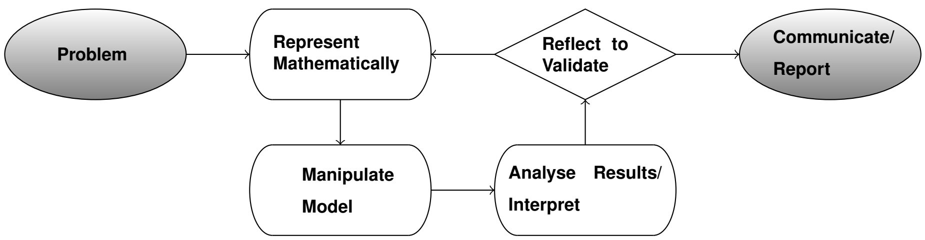
The Modeling Cycle

In high school, modeling involves a way of thought different from what students are taught when they learn much of the core K–8 mathematics. It provides experience in approaching problems that are not precisely formulated and for which there is not necessarily a single "correct" answer. Deciding what is left out of a model can be as important as deciding what is put in. Judgment, approximation, and critical thinking enter into the process. Modeling can have differing goals depending on the situation—sometimes the aim is quantitative prediction, for example in weather modeling, and sometimes the aim is to create a simple model that captures some qualitative aspect of the system with a goal of better understanding the system, for example modeling the cyclic nature of predator-prey populations.

Why is modeling difficult? Modeling requires multiple mental activities and significant human skills of abstraction, analysis, and communication. First, a real world situation must be understood in terms familiar to the student. Critical variables must be identified and those that represent essential features are selected. Second, the interpreted situation must be represented—by diagrams, graphs, equations, or tables. Moving from the interpretation to the representation involves reasoning—algebraic, proportional, quantitative, geometric, or statistical. Symbolic manipulation and calculation[A-SSE.3] may follow to produce expressions for the desired quantities. A critical step is now to interpret the quantitative information in terms of the original situation. The quantitative information must be analyzed or synthesized, that is, information is either combined to make

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

some judgment or separated into pieces to do so. During this analysis or synthesis, assumptions are either made or assumptions are evaluated. At this point, the information obtained is evaluated in terms of the original situation. If the information is unreasonable or inadequate, then the model may need to be modified to re-start the whole process. If the information is reasonable and adequate, the results are communicated in terms reflecting the original real world context and the information sought by the student. Understanding the limitations of the model involves critical thinking.



This figure is a variation of the figures in the introduction to high school modeling in the Standards.

Diagrams of modeling processes vary. For example, a diagram that focuses on reasoning processes has four components: Description, Manipulation, Translation or Prediction, and Verification.[5] Partitioning the modeling process into reasoning components is helpful in identifying where reasoning is succeeding or failing. This is important in both assessing student work and guiding instruction. These diagrams of modeling processes are intended as guides for teachers and curriculum developers rather than as illustrations of steps to be memorized by students.

Units and Modeling

Throughout the modeling process, units are critical for several reasons, including guiding the symbolic or numeric calculations.[N-Q.1] Keeping track of units is very helpful in determining if the calculations are meaningful and lead to the desired results. Units are also critical in the analysis and synthesis and in making or evaluating assumptions, as well as determining reasonableness of answer. For example, if analysis of a cost equation for driving an automobile indicates that a typical driver in the US will drive 5000 miles per year, one should check units to make sure that the gallons are US gallons and the fuel efficiency is in miles per US gallon. Most of the world measures gasoline in liters and distances in kilometers rather than miles. (According to the Federal Highway Administration, the average number of miles driven per year by US drivers is over 13,000.)

N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

[5] See Lesh & Doerr, 2003, *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching*, Lawrence Erlbaum Associates, p. 17.

Units are almost always essential in communicating the results of a model since answers to real world problems are usually quantities, that is, numbers with units. Modeling prior to high school produces measures of attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling such as acceleration, percent of GDP, person-hours, and some measures where the units are not specified and have to be understood in the way the measure is defined.[N-Q.2] For example, the S&P 500 stock index is a measure derived from the quotient of the value of 500 companies now and in 1940–42.

N-Q.2 Define appropriate quantities for the purpose of descriptive modeling.

Modeling and the Standards for Mathematical Practice

One of the eight mathematical practice standards—MP4 Model with mathematics—focuses on modeling and modeling draws on and develops all eight. This helps explain why modeling with mathematics and statistics is challenging. It is a capstone experience, the proof of the pudding. To embody this, students might complete a capstone experience in modeling.

Make sense of problems and persevere in solving them (MP1) begins with the essence of problem solving by modeling: "Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution." Solving a real life problem in a non-mathematical context by mathematizing (i.e. modeling) requires knowing the meaning of the problem and finding a mathematical representation. Later in this standard, "Younger students might rely on using concrete objects or pictures [i.e. models] to help conceptualize and solve a problem."

Reason abstractly and quantitatively (MP2) includes two critical modeling activities. The first is "the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate," and the second is that "Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved." Decontextualizing and representing are fundamental to problem solving by modeling.

Construct viable arguments and critique the reasoning of others (MP3) notes that mathematically proficient students "reason inductively about data, making plausible arguments that take into account the context from which the data arose"—the data being the model educed from some context. Further, "Elementary students can construct arguments using concrete referents (i.e. models) such as objects, drawings, diagrams, and actions." Discussing the validity of the model and the level of uncertainty in the results makes use of these skills.

Use appropriate tools strategically (MP5) notes that "When making mathematical models, [mathematically proficient students] know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data." Simulation provides an important path to explore the consequences of a model, and to see what happens when parameters of the model are varied.

Attend to precision (MP6). Here the most important consideration of modeling is to "express numerical answers with a degree of precision appropriate for the problem context" and in appropriate units. For example, if one is modeling the annual debt or surplus (there were no surpluses) in the US federal budget over the decade 2001–2010, then common options for a unit are nominal dollars, constant dollars, or percent of GDP. The degree of precision appropriate for understanding the model is to the nearest billion dollars (or nearest tenth percent of GDP) or perhaps the nearest ten billion dollars (or nearest percent of GDP).[N-Q.3] Beyond accuracy, modeling raises the issue of uncertainty—how likely are the quantities we want to model to be within a certain range. How much do features the model neglects affect accuracy and uncertainty?

N-Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Look for and make use of structure (MP7). Here, looking closely at a real world situation to discern relationships between quantities is critical for mathematical modeling. Students look for patterns or structure in the situation, for example, seeing the side of a right triangle when a shadow is cast by an upright flagpole as part of a right triangle or seeing the rise and run of a ramp on a staircase.

Look for and express regularity in repeated reasoning (MP8). Modeling activities often involve multistep calculations and the whole modeling cycle may need to be repeated. Here, mathematically proficient students "continually evaluate the reasonableness of their intermediate results" and "maintain oversight of the process" (in this case, the modeling process).

Modeling and Reasonableness of Answers

Continually evaluating reasonableness of intermediate results in problem solving is important in several of the standards for mathematical practice. Doing this often requires having reference values, sometimes called anchors or quantitative benchmarks, for comparison. Joel Best, in his book Stat-Spotting,[6] lists a few quantitative benchmarks necessary for understanding US social statistics: the US population, the annual birth and death rates, and the approximate fractions of the minority subpopulations. Without these reference values, an answer of 27 million 18-year-olds in the US population may seem reasonable. Such benchmarks for other measures are helpful, providing quick ways to mentally check intermediate answers while solving multistep problems. For example, it is very helpful to know that a kilogram is approximately 2 pounds, a meter is a bit longer than a yard, and there are about 3 liters in a gallon. This kind of quantitative awareness can be developed with prac-

[6] 2008, *Stat-Spotting: A Field Guide to Identifying Dubious Data*, University of California Press.

tice, and easily expanded with the immense amount of information readily available from the internet.

Statistics and Probability

Specific modeling standards appear throughout the high school standards indicated by a star symbol (★). About one in four of the standards in Number and Quantity, Algebra, Functions, and Geometry have a star, but the entire conceptual category of Statistics and Probability has a star. In statistics, students use statistical and probability models—whose data and variables are often embodied in graphs, tables, and diagrams—to understand reality. Statistical problem solving is an investigative process designed to understand variability and uncertainty in real life situations. Students formulate a question (anticipating variability), collect data (acknowledging variability), analyze data (accounting for variability), and interpret results (allowing for variability).[7] The final step is a report.

Much of the study of statistics and probability in Grades 6–8 concerns describing variability, building on experiences with categorical and measurement data in early grades (see the progressions for these domains). In high school the focus shifts to drawing inferences—that is, conclusions—from data in the face of statistical uncertainty. In this process, analyzing data may have two steps: representing data and fitting a function (often called the model) which is intended to capture a relationship of the variables. For example, bivariate quantitative data might be represented by a scatter plot and then the scatter plot is modeled as a linear, quadratic, or logarithmic function. A probability distribution might be represented as a bar graph and then the bar graph is modeled by an exponential function. See the high school Statistics and Probability Progression for examples.

Because the Statistics and Probability Progression for high school is also a modeling progression, the discussion here will only note statistics and probability standards when they are related to modeling standards in one of the other conceptual categories.

Developing High School Modeling

In early grades, students use models to represent addition and subtraction relationships among quantities such as 2 apples and 3 apples, and to understand numbers and arithmetic. Concrete models, drawings, numerical equations, and diagrams help to explain arithmetic as well as represent addition, subtraction, multiplication, and division situations described in the Operations and Algebraic Thinking Progression. Later, students use graphs and symbolic equations

[7] See the American Statistical Association's 2007 *Guidelines for Assessment and Instruction in Statistics Education*, Alexandria, VA: American Statistical Association, 2007, pp. 11–15, `http://www.amstat.org/education/gaise`.

to represent relationships among quantities such as the price of n apples where p is the price per apple. In Grade 8, calculating and interpreting the concept of slope may, in various contexts, draw on interpreting subtraction as measuring change or as comparison, and division as equal partition or as comparison (see Tables 2 and 3 of the Operations and Algebraic Thinking Progression). Creation of exponential models builds on initial understanding of positive integer exponents as a representation of repeated multiplication, while identifying the base of the exponential expression from a table requires the unknown factor interpretation of division. Extension of an exponential model from a geometric sequence to a function defined on the real numbers builds on the understanding of rational and irrational numbers developed in Grades 6–8 (see The Number System Progression).

By the beginning of high school, variables and algebraic expressions are available for representing quantities in a context. Modeling in high school can proceed in two ways. First, problems can focus directly on the concepts being studied, i.e., situations such as the path of a projectile which are modeled by quadratic equations can be a part of the study of quadratic equations. This is the traditional path followed by having a section of word problems at the end of a lesson. A second, more realistic, way to develop modeling is to utilize situations that can become more complex as more mathematics and statistics are learned.[8] It is unlikely that one situation can be used throughout high school modeling, but some situations can be increased in complexity (examples are given in this progression). Modeling with mathematics in high school begins with linear and exponential models and proceeds to representing more complex situations with quadratics and other polynomials, geometric and trigonometric models, logic models such as flow charts, diagrams with graphs and networks, composite functional models such as logistic ones, and combinations and systems of these. Modeling with statistics and probability (that is, as noted earlier, essentially all of statistics and probability) is detailed in the progression for that conceptual category.

Linear and Exponential Models

In high school, the most commonly occurring relationships are those modeled by linear and exponential functions. Examples abound. The number of miles traveled in t hours by an automobile at a speed of 30 miles per hour is $30t$ and the amount of money in an account earning 4% interest compounded annually after 3 years is $P(1.04)^3$ where P is the initial deposit. Students learn to identify the referents of

[8]For examples, see Schoen & Hirsch, "The Core-Plus Mathematics Project: Perspectives and Student Achievement," and Senk, "Effects of the UCSMP Secondary School Curriculum on Students' Achievement" in Senk & Thompson (Eds.), 2003, *Standards-Based School Mathematics Curricula: What Are They? What Do Students, Learn?*, Lawrence Erlbaum Associates.

symbols within expressions (MP2), e.g., 30 is the speed (or, later, velocity), t is the time in hours, and to abstract distance traveled as the product of velocity and time.

In Grade 8, students learned that functions are relationships where one quantity (output or dependent variable) is determined by another (input or independent variable).[8.F.1] In high school, they deepen their understanding of functions, learning that the set of inputs is the domain of the function and the set of outputs is the range.[F-IF.1] For example, the car traveling 30 miles per hour travels a distance d in t hours is expressed as a function

$$d(t) = 30t.$$

Students learn that when a function arises in a real world context a reasonable domain for the function is often determined by that context.

Students learn that functions provide ways of comparing quantities and making decisions. For example, a more fuel-efficient automobile costs \$3000 more than a less fuel efficient one, and \$500 per year will be saved on gasoline with the more efficient car. (This can be made precisely realistic by using data, say, from comparing a hybrid version to a gasoline version of an automotive model.) A graph of the net savings function

$$S(t) = 500t - 3000$$

(see margin) will have a vertical intercept at $S = -3000$ and a horizontal intercept at $t = 6$. Students learn that the horizontal intercept, or the zero of the function, is the break-even point, that is, by year 6 the \$3000 extra cost has been recovered in savings on gasoline costs.[F-LE.5]

As students learn more about comparing functions that have domains other than the nonnegative integers, this example can be increased in complexity.[10] The buyer has the option of paying the extra \$3000 and saving money on gasoline or placing the \$3000 in a savings account earning 4% per year compounded yearly. One option yields the net savings

$$S(t) = 500t - 3000$$

while the other yields amount

$$A(t) = 3000(1.04^t).$$

Students compare $S(t)$ and $A(t)$ by graphs or tables over some number of years, the domain of the functions.[F-IF.9] The expected time the buyer will drive the car determines a reasonable domain. A table of values for A and S (shown in the margin) over years 1 to 20 is likely to be sufficient for comparing the functions,[F-IF.6] or, later when

[10] Example from Madison, Boersma, Diefenderfer, & Dingman, 2009, *Case Studies for Quantitative Reasoning*, Pearson Custom Publishing.

8.F.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[9]

F-IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

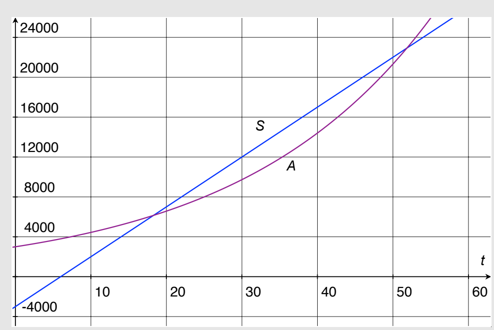
F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

Comparing Functions

Year t	$S(t)$	$A(t)$	Year t	$S(t)$	$A(t)$
0	-3000	3000	10	2000	4441
1	-2500	3120	11	2500	4618
2	-2000	3245	12	3000	4803
3	-1500	3375	13	3500	4995
4	-1000	3510	14	4000	5195
5	-500	3650	15	4500	5403
6	0	3796	16	5000	5619
7	500	3948	17	5500	5844
8	1000	4106	18	6000	6077
9	1500	4270	19	6500	6321

Outcomes for two scenarios at year t. If the hybrid is purchased, its savings on gasoline costs plus the difference in price between hybrid and gasoline models is shown as $S(t)$. If the gasoline model is purchased and the price difference is invested, the amount of the investment is $A(t)$.

Comparing Functions



Comparing outcomes for two scenarios: Buying and operating a hybrid automobile vs buying and operating a gasoline automobile and investing the difference in their prices.

F-IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

non-integer domains are understood, the graphs of S and A over the interval [0,20] will give considerable information (see the margin). The vertical intercepts of the two graphs and their two points of intersection are interpreted in the context of the problem. Analysis of the key features of the two graphs[F-IF.4] provides opportunities for students to compare the behaviors of linear and exponential functions. Students observe the average rates of change of the two functions over various intervals[F-IF.6] and see why the exponential function values will eventually overtake the linear function values and remain greater beyond some point. Students can now report on the information that will influence an economic decision by relating the behavior of the graphs to the comparative savings.

Students can again question the assumptions underlying the models of the two savings functions. What is the effect if the cost of gasoline changes? What is the effect if the number of miles driven changes? What will be the results of periodically (say, annually) placing the savings on gasoline costs in the savings account earning 4% per year compounded yearly? This latter option changes the linear model to a second exponential model, starts with a sum of a geometric series, which can be expressed either recursively or with an explicit formula,[F-BF.2] and points to the advantages of rewriting the sum of exponential expressions as a single exponential expression.[A-SSE.3c] This reinforces that algebraic re-writing of expressions is helpful, sometimes essential, to achieve comprehensible and usable models.

In the above example, students learn to question why the two scenarios have a \$6000 difference at year 0. Students might argue that the \$3000 is being invested two ways—one way is investing in the automobile and one way is placing in a savings account. The question then becomes: Which investment produces the most returns? That would make both functions be 0 at time 0. Is it more reasonable to note that the difference is \$3000 and not \$6000? In that case the graphs look like the ones here, and the table above is altered by reducing each entry for $A(t)$ by 3000.

Students learn that some initial representations and calculations can be done by hand,[F-IF.7] say, the graph of $S(t) = 500t - 3000$ and its key features. With iterations of the modeling cycle, the model becomes more complicated. Specific outputs of the functions can be calculated by hand, but technology is essential to understand the overall situation.

Students learn to distinguish between scenarios like the one above where two (or more) equations or functions give different results based on different assumptions about the situation and scenarios where the two (or more) equations (possibly, inequalities) or functions express relationships among the quantities of interest under the same assumptions. The latter scenarios are modeled by a system of equations or inequalities. A system of equations imposes multiple conditions on a situation, one for each of the equations. Solutions to systems must satisfy each of the equations. For example,

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

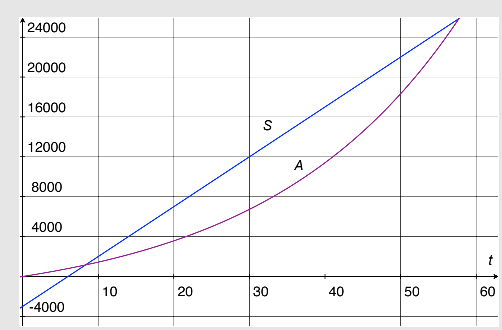
F-IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

F-BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

A-SSE.3c Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

c Use the properties of exponents to transform expressions for exponential functions.

Comparing Functions



F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a system of two linear equations[A-CED.3] will model the speed that you can row a boat with no current and the speed of the current provided you know the speed of the boat as you row with the current and the speed you can row against the current. Students learn how to describe situations by systems of two or three equations or inequalities and to solve the systems using graphs, substitution, or matrices. Students learn to detect if a system of equations is consistent, inconsistent, or independent.

Later, as students are challenged to develop more complex models, the processes of solving systems of equations are used to synthesize and develop new relationships from systems of equations that model a situation. Thus, students are challenged to use substitution to combine parametric equations and giving the spatial coordinates of a projectile as a function of time into a single relation modeling the path of the projectile, or to incorporate a constraint on the volume into a formula giving the cost of a cylindrical can as a function of the radius.

A-CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

Counting, Probability, Odds and Modeling

In Grades 7 and 8, students learned about probability and analysis of bivariate data. In high school, students learn the meanings of correlation and causation. Correlation, along with standard deviation, is interpreted in terms of a linear model of a data set. Students distinguish in models of real data the difference between correlation and causation.[S-ID.7, S-ID.8, S-ID.9]

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

S-ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.

S-ID.9 Distinguish between correlation and causation.

Students' intuitions, affected by media reports and the surrounding culture (cf. Nobel Laureate Daniel Kahnemann's Thinking Fast and Slow), sometimes conflict with their study of probability. Unusual events do occur and unconditional theoretical probabilities are based on what will happen over the long term and are not affected by the past—the probability of a head on a coin flip is $\frac{1}{2}$ even though each of the seven previous flips resulted in a head. Students learn how to reconcile accounts of probability from public and social media with their study of probability in school. For example, they learn the intriguing difference between conspiracy and coincidence.

Relating the study of probability to everyday language and feelings is important. Students learn about interpreting probabilities as "how surprised should we be?" Students learn to understand meanings of ordinary probabilistic words such as "unusual" by examples such as: "The really unusual day would be one where nothing unusual happens" and "280 times a day, a one-in-a-million shot is going to occur," given that there were approximately 280 million people in the US at the time. Coincidence is described as "unexpected connections that are both riveting and rattling."•

• "How surprised should we be?" is attributed to statistician Bradley Efron. "The really unusual day" and other examples are attributed to mathematician Persi Diaconis. See Belkin, 2002, "The Odds of That," New York Times Magazine, `http://www.nytimes.com/2002/08/11/magazine/11COINCIDENCE.html`.

Because probabilities are often stated in news media in terms of odds against an event occurring, students learn to move from probabilities to odds and back. For example, if the odds against a horse winning a race are 4 to 1, the probability that the horse will win is estimated to be $\frac{1}{1+4}$. If the probability that another horse will win is 0.4 then the odds against that horse winning is the probability of not winning, 0.6, to the probability of winning, 0.4, written as 0.6–0.4 or, equivalently, 3–2 or 3:2 and read as "3 to 2." The equivalence of 0.6–0.4 and 3–2 highlights the fact that odds are ratios of numbers, where the numerator and denominators convey meaning. Students learn that the sum of the probabilities of mutually exclusive events occurring cannot exceed 1, but that they sometimes do in media reports where odds and probabilities are approximated for simplicity.

Counting to determine probabilities continues into high school, and student learning is reinforced with models. For example, the birthday problem provides rich learning experiences and shows students some outcomes that are not intuitively obvious. Counting the number of possibilities for n birthdays yields an exponential expression 366^n, and counting of the number of possibilities for n birthdays all to be different yields a permutation P_n^{366}. The quotient is the probability that n randomly chosen people will all have different birthdays, yielding the probability of at least one birthday match among n people. The often surprising result that when $n = 23$ there is approximately a 50-50 chance (probability of 0.5 or 50–50 odds) of having a match. Students learn that the function

$$P(n) = \frac{P_n^{366}}{366^n}$$

models the probability of having no birthday match for n randomly chosen people, and $1 - P(n)$ is the probability of at least one birthday match. The results can be modeled by a spreadsheet revealing the probabilities for $n = 2$ to $n = 367$. Students learn that it requires at least 367 people to have a probability of 1 of a birthday match and also learn about the behavior of technology in that the spreadsheet values for the probability of at least one match become 1 (or at least report as 1) for values of n less than 367. Students calculating $P(n)$ using hand held calculators learn that for values of n of approximately 40, many hand-held calculators cannot compute the numerators and denominators because of their size. This provides an opportunity to learn that rewriting the quotient of the two, too large numbers as the product of a sequence of simpler quotients allows the calculator to compute the sequence of quotients and then take their product. On TI calculators this takes the form

$$\mathrm{Prod}\left(\mathrm{Seq}\left(\frac{x}{366}, x, 366, 366 - n + 1, -1\right)\right),$$

that is, the product of the sequence

$$\left\{\frac{366}{366}, \frac{365}{366}, \ldots, \frac{366 - n + 1}{366}\right\},$$

which students learn is a product of probabilities. Students learn that more complex questions can be asked about birthday matches.

For example, what is the probability of having exactly one, or exactly two matches of birthdays among n people?

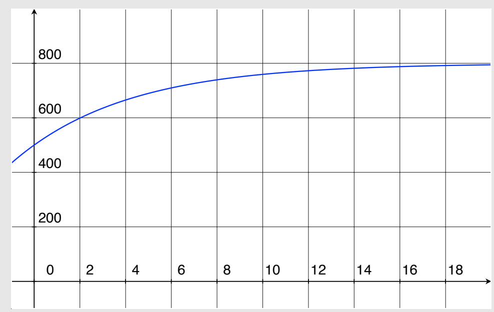
Key Features to Model

Students learn key features of the graphs of polynomials, rational functions, exponential and logarithmic functions, and modifications such as logistic functions to help in choosing a function that models a real life situation. For example, logistic functions are used in modeling how many students get a certain problem on a test right, and thereby are used in evaluating the difficulty of a problem on a standardized test. A quadratic function might be considered as model of profit from a business if the profit has one maximum (or minimum) over the domain of interest. An exponential function may model a population over some portion of the domain, but circumstances may constrain the growth over other portions. Piecewise functions are considered in situations where the behavior is different over different portions of the domain of interest.

Students learn that real life circumstances such as changes in populations are constrained by various conditions such as available food supply and diseases. They learn that these conditions prevent populations from growing exponentially over long periods of time. A common model for growth of a population P results from a rate of change of P being proportional to the difference between a limiting constant and P, as in Newton's law of cooling. This constrained exponential growth results in P being given by the difference between the limiting constant and an exponentially decaying function. For example, the margin shows the graph of a population that is initially 500 and approaches a limiting value of 800. Another common population growth model results from logistic functions where the rate of growth of P is proportional to the product of $P(a-P)$ for some constant a.

Students learn to look at key features of the graphs of models of constrained exponential growth (or decay) and logistic functions (intercepts, limiting values, and inflection points) and interpret these key features into the circumstances being modeled.[F-IF.4]

A common model for population growth



The function shown in the graph is $P(t) = 800 - 300e^{-kt}$ *where* k *is a positive constant, the solution to* $\frac{dP(t)}{dt} = k(800 - P(t))$.

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Formulas as Models

Formulas are mathematical models of relationships among quantities. Some are statements of laws of nature—e.g., Ohm's Law, $V = IR$, or Newton's law of cooling—and some are measurements of one quantity in terms of others—e.g., $V = \pi r^2 h$, the volume of a right circular cylinder in terms of its radius and height.[G-MG.2] Students learn how to manipulate formulas to isolate a quantity of interest. For example, if the question is to what depth will 50 cubic feet of a garden mulch cover a bed of area 20 square feet, then the formula $h = \frac{V}{A}$ where h is the depth, V is the volume, and A is the area, is an appropriate form.[A-CED.4] If one wants a depth of 6

G-MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

A-CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

inches, then the form would be the appropriate for finding how much mulch to buy. Students learn that the shape of the bed (modeled as the base of a cylinder) does not matter, an application of Cavalieri's principle;[G-GMD.1] volume is the product of the area and the height.[G-GMD.3]

G-GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.

G-GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Formulas that are models may sometimes be readily transformed into functions that are models. For example, the formula for the volume of a cylinder can be viewed as giving volume as a function of area of the base and the height, or, rearranging, giving the area of the base as a function of the volume and height. Similarly, Ohm's law can be viewed as giving voltage as a function of current and resistance. Newton's law of cooling states that the rate of change of the temperature of a cooling body is directionally proportional to the difference between the temperature of the body and the temperature of the environment, i.e., the ambient temperature.[F-BF.1b] This is another example of constrained exponential growth (or decay). The solution of this change equation (a differential equation) gives the temperature of the cooling body as a function of time.

F-BF.1b Write a function that describes a relationship between two quantities.

- b Combine standard function types using arithmetic operations.

In Grade 7 students learned about proportional relationships and constants of proportionality.[7.RP.2] These surface often in high school modeling. Students learn that many modeling situations begin with a statement like Ohm's law or Newton's law of cooling, that is, that a quantity of interest, I, is directly proportional to a quantity, V, and inversely proportional to a quantity, R, i. e. I is given by the product of a constant and $\frac{V}{I}$. Newton's law of cooling is stated as a proportionality giving the rate of change of the temperature at a given moment as a product of a constant and the difference in the temperatures—this can be used in forensic science to estimate the time of death of a murder victim based on the temperature of the body when it is found.

7.RP.2 Recognize and represent proportional relationships between quantities.

Right Triangle and Trigonometric Models Students learn that many real world situations can be modeled by right triangles. These include areas of regions that are made up of polygons, indirect measurement problems, and approximations of areas of non-polygonal regions such as circles. Examples are areas of regular polygons, height of a flag pole, and approximation of the area of a circle by regular polygons. Prior to extending the domains of the trigonometric functions by defining them in terms of arc length on the unit circle, students understand the trigonometric functions as ratios of sides of right triangles. These functions, paired with the Pythagorean Theorem, provide powerful tools for modeling many situations.[G-SRT.8]

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

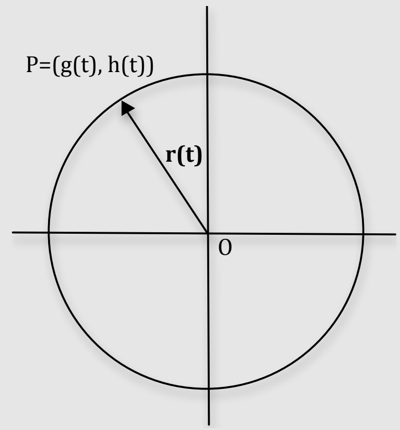
When the domains of the trigonometric functions are extended beyond acute angles,[F-TF.2] the reasons that these functions are called "circular functions" become clearer. Many situations involving circular motion can be modeled by trigonometric functions. The example below uses trigonometric functions and vector-valued functions. For example, prior to GPSs, this is how sailors determined their latitude.

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Where the Modeling Progression might lead

As mentioned earlier, modeling in high school becomes more complex and powerful as more mathematics and statistics are used to describe real life circumstances. As students learn more, they learn to use new concepts to extend simpler models previously studied. Although a high school modeling problem is not likely to incorporate all of high school mathematics, there are models that incorporate many concepts and extend beyond the high school mathematics described in the Standards. The motion of communication satellites around the earth or the motion of an object spinning rapidly in a circle by holding one end of a string with the other attached to the object can be modeled as a point traversing a circle. The object (at point P) is accelerated toward the center (O) of the circular path and the magnitude of the acceleration is constant.

The position vector $\vec{r}(t)$ joining O and P at time t is given by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$ where $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$ are unit vectors. By considering the geometry and the physics of the situation, one can show that there are functions $g(t)$ and $h(t)$ (twice differentiable, giving the velocity and acceleration vectors of the motion) satisfying the conditions of the model. Noting the similarities of the conditions on g and h to the behavior of the trigonometric functions $\sin(t)$ and $\cos(t)$ one can show that indeed the vector function describes uniform circular motion for an object P on a circle of radius R and a constant magnitude of acceleration.[F-TF.5]



Adapted from Usiskin, Peressini, Marchisotto, & Stanley, 2003, Mathematics for High School Teachers: An Advanced Perspective, *Pearson Prentice Hall, pp. 469–474.*

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.