Progressions for the Common Core State Standards in Mathematics (draft)

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For discussion of the Progressions and related topics, see the Tools for the Common Core blog: **http://commoncoretools.me**.

High School, Quantity

Overview

In high school, students in high school extend their conceptions of unit and of quantity. They encounter situations in which they must conceptualize relevant attributes and create or choose suitable measures for them. This work builds on students' previous experiences with units and systems of units in grades 1–8. Thinking about units is important not only in the domain of Measurement and Data, but also for students' work with multi-digit numbers (Number and Operations in Base Ten), and with fractions and unit fractions (Number and Operations—Fractions). In grades 6–8, students use units in working with numerical data (Statistics and Probability), with ratios and rates (Ratios and Proportional Relationships), with scientific notation (Expressions and Equations), and with measurements of angles, lengths, areas, and volumes (Geometry).

In the high school Standards, individual modeling standards are indicated by a star symbol (★). Because of their strong connection with modeling, the standards listed under Quantities are starred, indicating that all of these standards are modeling standards. Although the Vector and Matrix Quantities standards are not starred, they describe skills that, once learned, are often used in applied contexts. Among them are applications in physics and engineering. All of the Vector and Matrix Quantities standards have a plus sign, indicating that they involve mathematics that students should learn in order to take advanced mathematics courses.[1]

Brief tasks targeting the specific skills enumerated in the Quantities standards are possible and this progression includes some examples, but the skills described in these standards would also

This progression discusses the Number and Quantity standards that concern quantity. The remaining Number and Quantity standards are discussed in the Number Progression.

[1]The high school standards specify the mathematics that all students should study in order to be college and career ready. Additional material corresponding to (+) standards, mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics, is indicated by a plus sign in the left margin. Note, however, that not all (+) standards are prerequisites for every advanced mathematics course. For example, knowledge of vectors and matrices is not a necessary prerequisite for a single-variable calculus course.

be naturally prominent in the context of more substantial applied problems, lab reports in science classes, or even research papers for courses in social studies and technical subjects. Students' work with quantities can foster strong connections between mathematics and other subjects.

Quantities, derived quantities, and derived units Quantity is an integral part of any application of mathematics, and has connections to solving problems using data, equations, functions, and modeling. In the Standards, a quantity is a measurement that can be specified using a number and a unit. For example, 2.7 centimeters or the distance from the earth to the moon in miles are both quantities that involve measurements of the attribute length. Descriptors without units are sometimes also called quantities, e.g., the distance from the earth to the moon is called a quantity. In this case, a unit of measurement is not given, but can be chosen. However, numerical values without units are not quantities: 2.7 centimeters is a quantity, but 2.7 is not. The numerical value alone does not indicate what attribute is being measured, so no unit of measurement can be chosen.

It can make sense to add two quantities, such as when a child 51 inches tall grows 3 inches to become 54 inches tall. To be added or subtracted, quantities must be measurements of the same attribute (length, area, speed, etc.) and expressed in the same units. Converting quantities expressed in different units to have the same units is like converting fractions to have a common denominator before adding or subtracting. But, even when quantities have the same units it does not always make sense to add them. For example, if a wooded park with 300 trees per acre is next to a field with 30 trees per acre, they do not have 330 trees per acre.

Two or more quantities can give rise to new types of quantities, called derived quantities. These are sometimes described as products or quotients of attributes or units, for example: speed (distance divided by time), rectangular area (length multiplied by length), density (mass divided by volume), or population density (number of people divided by land area). For those familiar with derived quantities, these descriptions are useful shorthand. However, they may suggest that a derived quantity is written as a product or quotient of other quantities, or is not itself a quantity. Like all quantities, derived quantities can be expressed as a number followed by a unit. Understanding such derived quantities requires students to understand two or more quantities simultaneously (e.g., speed as entailing displacement *and* time, simultaneously).

Before high school, the derived quantities that students encounter include area, volume, and examples of rates. In their work with area and volume, students use derived units and their abbreviations, e.g., sq cm and cu cm (see the Geometric Measurement Progression), but use of such abbreviations is not a focus of their work with rates.

For instance, they consider examples such as: if Sharoya walks 3 meters every 2 seconds, she walks at a rate of $\frac{3}{2}$ meters per second (see the Ratio and Proportional Relationships Progression). In high school, students view such rates more abstractly and abbreviate derived units, e.g., writing meters per second as m/s. Moreover, they become more sophisticated in their use of derived units, recognizing when it is necessary to convert between different units for speed and other derived quantities. In measuring and in computations with measurements, they choose appropriate units and levels of accuracy for measurements of familiar attributes. When investigating novel situations, they identify quantifiable features of interest and units in which to measure them.

Depending on context, quantities are called by different names, such as "measure" (e.g., productivity measure) or "index" (e.g., Consumer Price Index). In situations where quantities are represented as variables, quantities are often referred to as "variables," eliding the distinction between a quantity and its representation as a variable. These subtleties in terminology do not need to be made explicit to students, but students need to use terms correctly in context.

Quantities

Units are central to applied mathematics and everyday life. Units are prominent in a wide variety of situations involving, for example, acceleration, currency and currency conversions, people-hours, energy and power, concentration and density, social science rates (e.g., number of deaths per 100,000), and other everyday rates (e.g., points per game).

Using and interpreting units Reasoning quantitatively includes knowing when and how to convert units in computations, such as when adding and subtracting quantities that measure the same attribute but are expressed in different units and other computations with measurements in different units or converting units for derived quantities such as density and speed. Reasoning quantitatively can also include analyzing the units in a calculation to reveal the units of the answer. This can to help reveal a mistake if, for example, the answer comes out to be a distance when it should be a speed (MP.2).

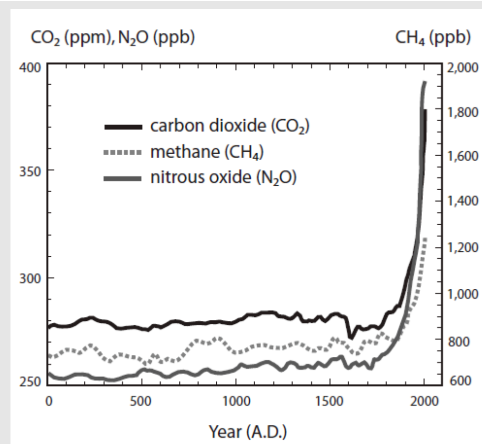
In applications, formulas are often used, and errors can occur in the use of the formulas if units are not attended to carefully.[N-Q.1] The formula $d = vt$ notwithstanding, a car driving at 25 mph for 3 minutes does not cover a distance of 25×3 miles. Conversely, if the student *does* attend carefully to units, the result can be a deeper understanding of a formula or a situation.[N-Q.1] Students should specify units when defining variables and attend to units when writing expressions and equations (MP.6).

A good quantitative understanding of the situation at hand helps a student make sound choices for the scale and origin of a graph or a a display.[N-Q.1] In a map of arable land area, for example, there is no sense in having a scale that extends to negative values; in a graph showing the concentration of atmospheric carbon dioxide over the past 2000 years, the choice of origin in the vertical scale is an important editorial decision (see figure in the margin). These considerations apply to graphs, data tables, scatter plots, and other visual displays of numerical data. It should go without saying that graphs and displays must be properly labeled, or else they are meaningless (MP.6).

Level of accuracy Quantitative reasoning includes choosing an appropriate level of accuracy when reporting quantities.[N-Q.3] For example, if the doctor measures your height as 73 inches and your weight as 210 pounds, then your Body Mass Index (BMI) is

$$\frac{\text{(weight in pounds)}}{\text{(height in inches)}^2} \times 703 = \frac{210}{73^2} \times 703 \approx 27.7031 \approx 28.$$

N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.



Concentration of carbon dioxide and other gases in the atmosphere over the past 2,000 years. Source: Forster et al., 2007, Changes in Atmospheric Constituents and in Radiative Forcing. In Solomon et al. (Eds.), *Climate Change 2007: The Physical Science Basis*, Figure 1, p. 135, `http://www.ipcc.ch/pdf/assessment-report/ar4/wg1/ar4-wg1-chapter2.pdf`.

N-Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

There is no point in reporting a value more precise than 28 here, because any value between 25 and 30 is considered overweight.•

• See http://en.wikipedia.org/wiki/Body_mass_index. Moreover, your weight varies somewhat from week to week. (For that matter, the 703 in the formula is itself an approximation; it is a conversion factor between the International System of Units (SI) and English units, with a precise value of 703.0695 . . . ; again, the digits to the right of the decimal point are unnecessary.)

Defining quantities In modeling situations (MP.4), defining the key quantity of interest might be part of the task. For example, in a situation that involves crop productivity, a student might him- or herself choose to examine the number of tons of fertilizer per acre as the variable of interest. In a situation that involves content development for a web site, a choice might arise as to whether the number of posts per day or the number of words per day is the key productivity variable. (For other examples of variables, see the section on units and modeling in the Modeling Progression. Different choices for the quantity of interest may result in different models, as illustrated by models for oak tree growth in the Statistics and Probability Progression.) Such decisions about determining and describing the quantity of interest are sometimes left to the student in modeling tasks.[N-Q.2] Tasks of this kind may be most effective if students have a genuine question that leads them to quantify features of the situation being analyzed.

N-Q.2 Define appropriate quantities for the purpose of descriptive modeling.

Vector and Matrix Quantities

As with many other concepts in the Standards, the concepts of vector and matrix quantities generalize familiar ideas and extend familiar representations. In this case, the ideas generalized involve number, magnitude, and operations.

Vectors In early grades, students start on the path toward understanding real numbers as points on the number line. They began by using numbers to count, then in measuring lengths by iterating a unit.[1.MD.2] Seeing whole numbers as concatenations of unit lengths yields a correspondence between numbers and points on a number line diagram.[2.MD.6] Sums of whole numbers are represented on the number line as concatenations of two line segments. Partitioning the unit into equal pieces allows fractions and finite decimals to be represented as line segments,[3.NF.2] and their sums to be represented as concatenations of these segments.

For negative numbers, the correspondence between numbers and the number line needs to attend to direction.

Now the line segments have directions, and therefore a beginning and an end. When concatenating these directed line segments, we start the second line segment at the end of the first one. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively cancelling part or all of it out.[7.NS.1b] As students encounter vectors, they will be able to see their previous work with adding numbers as adding one-dimensional vectors, and their previous work with multiplying numbers as scalar multiplication of one-dimensional vectors. In the illustrations of addition on the number line shown in the Number System Progression, it is implicit that a real number is represented by *any* directed line segment that is parallel to the number line with the appropriate orientation (indicated by an arrow at the terminal point) and magnitude (indicated by length of the segment).

Vectors in the plane maintain these conventions. An arrow indicates orientation and length indicates magnitude.• Any two parallel line segments of the same orientation and length represent the same vector.[N-VM.1]

As on the number line, the sum of two vectors $\mathbf{v}+\mathbf{w}$ can be shown in the plane by choosing representations so that the terminal point of $\mathbf{v}$ is the initial point of $\mathbf{w}$. Their sum is represented as a third vector that has the initial point of $\mathbf{v}$ and the terminal point of $\mathbf{w}$. If $\mathbf{v}$ and $\mathbf{w}$ are not parallel, their sum can be seen as the remaining side of a triangle formed by $\mathbf{v}$ and $\mathbf{w}$, showing graphically that

$$||\mathbf{v}+\mathbf{w}|| < ||\mathbf{v}|| + ||\mathbf{w}||.$$

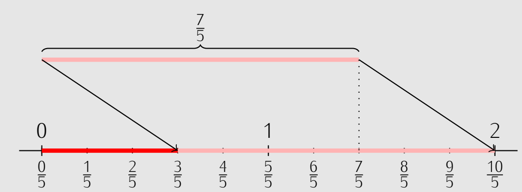
With auxiliary line segments (MP.7), the sum can also be seen as the diagonal of a parallelogram. Because any two parallel line segments with the same orientation and length represent the same

[1.MD.2] Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

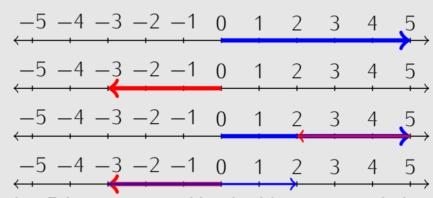
[3.NF.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

Representing addition as concatenation on the number line



[7.NS.1b] Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

Showing $5 + (-3) = 2$ and $-3 + 5 = 2$ on the number line



The number 5 is represented by the blue arrow pointing right from 0, and 3 is represented by the red arrow pointing left. To compute $5 + 3$, place the arrow for 5, then attach the arrow for 3 to its endpoint. To compute $-3 + 5$, place the arrow for -3, then attach the arrow for 3 to its endpoint.

• Vectors are often written in boldface (e.g., $\mathbf{v}$) or with an arrow (e.g., $\vec{v}$). Notation for magnitudes is sometimes the same or similar to that for absolute value (e.g., $|\mathbf{v}|$ or $||\mathbf{v}||$) or puts the letter representing the vector in italics (e.g., v). Each of these has various advantages and disadvantages. For example, the italic notation does not lend itself to expressing $||c\mathbf{v}||$, however, when usable, indicates at a glance that the object represented is not a vector, but a real number, suggesting that algebraic techniques can be used. A disadvantage of $|\mathbf{v}|$ is that it might be confused with absolute value, thus its use is often discouraged.

[N-VM.1] (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $||\mathbf{v}||$, v).

vector, the sides of this parallelogram can be used to represent $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ in different ways. The two addends can be shown as sharing an initial point, illustrating the parallelogram rule for vector addition.[N-VM.4a] Or they can be shown so that the terminal point of $\mathbf{w}$ is the initial point of $\mathbf{v}$ (illustrating commutativity of vector addition).

When students began to use negative numbers, they learned to understand subtraction as adding the additive inverse (7.NS.1c). Similarly, they understand vector subtraction as adding the (vector) additive inverse.[N-VM.4.c] As on the number line, the additive inverse $-\mathbf{w}$ of a vector $\mathbf{w}$ is represented as a directed line segment with an orientation opposite to that of $\mathbf{w}$. If augmented by its remaining diagonal (MP.7), the parallelogram used to illustrate the sum $\mathbf{v}+\mathbf{w}$ can be recycled to illustrate the difference $\mathbf{v}-\mathbf{w}$.[G-SRT.1]

Like vector addition and subtraction, multiplying a vector by a scalar builds on ideas and representations from earlier grades. When students began extending multiplication to fractions, visualizing a product as a concatenation of directed line segments by thinking of multiplication as repeated addition begins to break down. Interpreting multiplication as scaling[5.NF.5] rather than repeated addition maintains a way to visualize the correspondence between products of two numbers and their representations on the number line. This correspondence has a geometric interpretation as dilation of a line segment. The center of the dilation is the origin of the number line. Depending on the scale factor, the image of the segment is longer or shorter, but remains a segment on the number line.[G-SRT.1] In terms of a vector $\mathbf{v}$ and scalar c, this is $||c\mathbf{v}|| = |c| \cdot ||\mathbf{v}||$. When students learned to multiply by -1, they represented the corresponding effect on a directed line segment as reversing its orientation. They continue to do so when the directed line segment represents a vector in the plane.[N-VM.2]

Coordinates for the plane allow the results of operations on vectors to be computed symbolically, in terms of x- and y-components or in terms of magnitude and direction. Students compute the components of vectors from their initial and terminal points in rectangular coordinates,[N-VM.2] and use components to compute sums, differences, scalar multiples, and magnitudes.[N-VM.4ac, N-VM.5a] Students compute sums and scalar multiples of vectors described in terms of magnitude and direction.[N-VM.4b, N-VM.5b] Graphical representations of operations and magnitudes of vectors together with the different ways to compute them symbolically provide correspondences to be identified and explained (MP.1).

Matrices An understanding of matrices and operations on matrices might begin with similarity transformations• on vectors in the coordinate plane, building on earlier work in Grade 8 and high school. In Grade 8, students work with physical models and software to understand the effects of rigid motions and dilations on two-dimensional objects, describing these effects in terms of coordinates[8.G.3] and un-

[N-VM.4](+) Add and subtract vectors.

a Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c Understand vector subtraction $\mathbf{v}-\mathbf{w}$ as $\mathbf{v}+(-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

[G-SRT.1] Verify experimentally the properties of dilations given by a center and a scale factor:

[5.NF.5]Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number . . . .

[N-VM.2](+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

[N-VM.5](+) Multiply a vector by a scalar.

a Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.

b Compute the magnitude of a scalar multiple $c\mathbf{v}$ using $||c\mathbf{v}|| = |c|v$. Compute the direction of $c\mathbf{v}$ knowing that when $|c|v \neq 0$, the direction of $c\mathbf{v}$ is either along $\mathbf{v}$ (for $c > 0$) or against $\mathbf{v}$ (for $c < 0$).

• Similarity transformations are rigid motions followed by dilations.

[8.G.3]Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

derstanding that two of these objects are congruent if one can be carried onto the other by a sequence of rigid motions. In high school, students experiment with similarity and other transformations in the plane, describing their effects in geometrical terms.[G-CO.2, G-CO.5] Students can use their high school understanding of functions and composition of functions (see the Functions Progression) to view sequences of transformations (such as the sequences of rigid motions from Grade 8) as compositions of transformations. With coordinates for the plane, rigid motions, dilations, and other (but not all) transformations can be represented by 2×2 matrices. Compositions of these transformations can be computed symbolically as products of the matrices that represent them. When a vector in the plane is represented as a column matrix whose entries are its components, multiplication of the column matrix by a 2×2 matrix can be interpreted as a transformation of the vector.[N-VM.11] Connections among vectors, transformations, and matrices can be analyzed further. For example, if two vectors in the plane are not parallel, then they and their sum can be seen as sides of a triangle. How does the transformation determined by a given 2×2 matrix affect this triangle? Students might choose special vectors to analyze (MP.1) such as (1,0) and (0,1), helping them to make use of structure (MP.7) in interpreting the absolute value of the determinant in terms of area,[N-VM.12] and in thinking about when the transformation has an inverse.[N-VM.10]

Students use matrices in other ways, for example, to represent the Hot Potato payoff described in the high school Statistics and Probability Progression, and doubling of that payoff.[N-VM.6, N-VM.7]

G-CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

N-VM.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.12 (+) Work with 2×2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.

N-VM.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.6 (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

Where the Quantity Progression might lead

A wide variety of different units and methods of displaying them (well or poorly) occur in different disciplines. Some aspects of quantitative literacy involve choice of scale and units in data displays. In advanced modeling exercises, students can identify quantities relevant to a situation and use the units of those quantities to generate conjectures about algebraic relationships among them. This method, called dimensional analysis, can be used for example to determine that the period of a pendulum is independent of its mass. Vectors can represent quantities that change over time. For example, the position of a satellite in orbit around the Earth at any given moment can be represented as a vector, so the satellite's position over time can be represented as a vector-valued function. The trajectory in space that is traced out by the satellite can be represented as a parametric curve (see the end of the Modeling Progression).

Appendix: Brief Examples for N-Q.1

1. A textbook printed the following formula for the surface area of a cylinder:
$$SA = 2\pi r + 2\pi rh.$$
Kim had never studied geometry, but she knew there must be a typographical error in this formula. How could she tell?

2. The Trans-Alaska Pipeline System is 800 miles long and cost $8 billion to build. Divide one of these numbers by the other. What is the meaning of the answer?

3. Greenland has a population of 56,700 and a land area of 2,175,600 square kilometers. By what factor is the population density of the United States, 80 persons per square mile, larger than the population density of Greenland?

4.† A doctor orders Ceclor elixir for a child who weighs 9.3 kg. The child must receive 25 mg of the drug for each kilogram of body weight. The hospital pharmacy stocks Ceclor elixir in a concentration of 250 mg per 5 ml. How many milliliters of the stock elixir should the child receive?

   a) Estimate the answer mentally. (Suggestion: approximate the child's weight as 10 kg.)
   b) Compute the answer to the nearest tenth of a milliliter.

5.‡ A liquid weed-killer comes in four different kinds of bottles. The table in the margin gives information about the concentration, size, and price of the bottles. The "concentration" refers to the percentage of "active ingredient" in the bottle. The rest of the liquid in the bottle is water. For example, to calculate the amounts in Bottle B: $(0.18)(32)$ is 5.76, so there are 5.76 fluid ounces of active ingredient; $32 - 5.76 = 26.24$, so there are 26.24 fluid ounces of water.

   a) Rank the four bottles in order of how good a buy each represents. State what criterion you are using.
   b) Suppose a job calls for a total of 12 fluid ounces of active ingredient. How much would you need to spend if you bought Type A bottles? Type B bottles? Type C bottles? Type D bottles?

Different kinds of weed-killer

	concentration	amount in bottle	price of bottle
A	0.96%	64 fl oz	$12.99
B	18.0%	32 fl oz	$29.99
C	41.0%	32 fl oz	$39.99
D	0.96%	24 fl oz	$5.99

6. The distance traveled by a freely falling object dropped from rest is given by the formula
$$s = \frac{1}{2}gt^2.$$
Here s is the distance fallen, g is a constant representing the force of gravity at Earth's surface, and t is the duration of time over which the object falls. If s has units of meters and t has units of seconds, what must be the units of g? If we interpret g as a rate of change, what sort of quantity is changing with time?

7. A table in a construction manual lists the "k-values" of different building materials. The k-value measures how easily heat flows through a material. The k-value of concrete is given as
$$2.5\frac{\text{BTU} \cdot \text{in}}{\text{hr} \cdot \text{ft}^2 \cdot {}^\circ\text{F}}.$$
A BTU is a unit of heat energy. The construction manual gives the following example problem illustrating the k-value: How many BTUs of heat energy would be lost through a 100 ft^2 concrete wall 6 inches thick over a 12 hour period, if the temperature difference from one side of the wall to the other is 70°F? Your friend Anders doesn't know much about construction—or about heat loss—yet he was able to get the answer, 35,000 BTUs, just by thinking about units. How did he get the answer?

†Adapted from *Ready or Not: Creating a High School Diploma That Counts*. Achieve, 2004.

‡This task is due to Dick Stanley.